Quantum Electrodynamics

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February 21, 2005

Abstract

These special lectures were delivered in 1973, while I was a teacher of physics at the Illinois Institute of Technology in Chicago.

The attempt to reconcile the special theory of relativity (1905) and quantum mechanics (1926) led to the Klein-Gordon equation and the Dirac equation as possible replacements for the nonrelativistic Schrödinger wave equation. Although the Dirac equation appeared to provide a more complete description of the hydrogen atom spectrum than did the Schrödinger equation, it soon became evident that what was really needed was a theory that could describe creation and annihilation of particles. As a result, quantum electrodynamics was born. This theory not only provided an extremely accurate description of the interaction of photons, electrons and the newly discovered positrons, it also provided a fruitful model for describing the interaction of the other more massive particles that were soon to be discovered in the realm of high energy physics.

1 Noninteracting Bosons and Fermions

The photon is a massless particle of spin 1, a boson, while electrons and positrons are particles of spin 1/2, fermions. Before launching into a study of their interaction, we shall consider noninteracting bosons and fermions.

Bosons

Consider a Bose particle (i.e., integral spin particle) with a definite spin component r and a definite momentum $\hbar\vec{k}$. In the absence of interactions, the energy of this particle is simply

$$\hbar c\sqrt{|\vec{k}|^2 + \left(\frac{mc}{\hbar}\right)^2}.$$

If there are n such particles, the total momentum would be $n\hbar\vec{k}$, while the total energy would be

$$n\hbar c\sqrt{|\vec{k}|^2 + \left(\frac{mc}{\hbar}\right)^2}.$$

Thus, *for each possible value of* $\vec{k}$ one gets a set of equally spaced energy levels reminiscent of the level scheme of a simple harmonic oscillator in non-relativistic quantum mechanics.

Drawing upon this analogy with the simple harmonic oscillator, we are prompted to introduce the energy-momentum operator

$$P^\mu = \int d^3k\ \hbar k^\mu C^*(\vec{k})C(\vec{k}), \tag{1}$$

where $C^*(\vec{k})$ and $C(\vec{k})$ are, respectively, *creation* and *annihilation* operators such that

$$[C(\vec{k}), C(\vec{k}')] = 0 \tag{2}$$

and

$$[C(\vec{k}), C^*(\vec{k}')] = \delta(\vec{k} - \vec{k}'). \tag{3}$$

With these definitions it follows immediately that

$$[P^\mu, C^*(\vec{k})] = \hbar k^\mu C^*(\vec{k}) \tag{4}$$

and

$$[P^\mu, C(\vec{k})] = -\hbar k^\mu C(\vec{k}). \tag{5}$$

These commutation relations are *essential* if $C^*(\vec{k})$ and $C(\vec{k})$ are to be identified as operators which create and destroy, respectively, a particle of momentum $\hbar\vec{k}$ and energy

$$\hbar c\sqrt{|\vec{k}|^2 + \left(\frac{mc}{\hbar}\right)^2}.$$

The *vacuum state* $|\Omega>$ is that eigenstate of P^μ corresponding to eigenvalue 0 which satisfies

$$C(\vec{k})|\Omega >= 0 \tag{6}$$

for all values of $\vec{k}$. This state $|\Omega>$ is the "ground state" of quantum field theory.

The "excited states" of quantum field theory are generated by applying creation operators to $|\Omega>$. For example,

$$|\vec{k}>= C^*(\vec{k})|\Omega> \tag{7}$$

is a single particle state with momentum $\hbar\vec{k}$ and energy

$$\hbar c\sqrt{|\vec{k}|^2+\left(\frac{mc}{\hbar}\right)^2}.$$

This can be seen from the fact that

$$[P^\mu, C^*(\vec{k})]|\Omega>= \hbar k^\mu C^*(\vec{k})|\Omega>, \tag{8}$$

which is equivalent to

$$P^\mu[C^*(\vec{k})|\Omega>] = \hbar k^\mu[C^*(\vec{k})|\Omega>], \tag{9}$$

since $P^\mu|\Omega>= 0$.

This single particle state $|\vec{k}>= C^*(\vec{k})|\Omega>$ is normalized so that

$$<\vec{k}|\vec{k}'>=<\Omega|C(\vec{k})C^*(\vec{k}')|\Omega>=<\Omega|[C(\vec{k}), C^*(\vec{k}')]|\Omega>= \delta(\vec{k}-\vec{k}')<\Omega|\Omega>. \tag{10}$$

Thus, if $|\Omega>$ is normalized so that

$$<\Omega|\Omega>= 1, \tag{11}$$

then the single particle state is normalized so that

$$<\vec{k}|\vec{k}'>= \delta(\vec{k}-\vec{k}'). \tag{12}$$

A two particle state

$$|\vec{k}_1, \vec{k}_2>= \frac{1}{\sqrt{2}}C^*(\vec{k}_1)C^*(\vec{k}_2)|\Omega> \tag{13}$$

may be formed by repeated use of creation operators. The state vectors formed in this manner are *automatically* symmetric under interchange of the momenta. For example,

$$|\vec{k}_1, \vec{k}_2>= +|\vec{k}_2, \vec{k}_1>. \tag{14}$$

We have chosen the normalization constant so that

$$<\vec{k}_1, \vec{k}_2|\vec{k}_1', \vec{k}_2'>= \frac{1}{2}\left[\delta(\vec{k}_1-\vec{k}_1')\delta(\vec{k}_2-\vec{k}_2')+\delta(\vec{k}_1-\vec{k}_2')\delta(\vec{k}_2-\vec{k}_1')\right]. \tag{15}$$

Fermions

Consider now a Fermi particle (i.e., half-integral spin particle) subject to the Pauli exclusion principle. For the energy-momentum operator P^μ we retain the expression

$$P^\mu = \int d^3k\ \hbar k^\mu C^*(\vec{k})C(\vec{k}), \tag{16}$$

but we now suppose that $C^*(\vec{k})$ and $C(\vec{k})$ satisfy *anti-commutation relations*,

$$\{C(\vec{k}), C^*(\vec{k}')\} = 0 \tag{17}$$

and

$$\{C(\vec{k}), C^*(\vec{k}')\} = \delta(\vec{k} - \vec{k}'). \tag{18}$$

The "anti-commutator" is $\{A, B\} = AB + BA$, while the "commutator" is $[A, B] = AB - BA$. It may be verified that the essential relations

$$[P^\mu . C^*(\vec{k})] = \hbar k^\mu C^*(\vec{k}) \tag{19}$$

and

$$[P^\mu . C(\vec{k})] = -\hbar k^\mu C(\vec{k}) \tag{20}$$

are still satisfied, so $C^*(\vec{k})$ and $C(\vec{k})$ still have the character of creation and annihilation operators.

The vacuum state $|\Omega>$ is still defined by the condition

$$C(\vec{k})|\Omega >= 0 \tag{21}$$

for all $\vec{k}$, and the single particle states are still defined by Eq. (7) and the multiple particle states are still constructed by repeated application of creation operators, as in Eq. (13). However, because of the anti-commutation relations, we now have

$$|\vec{k}_1, \vec{k}_2 >= -|\vec{k}_2, \vec{k}_1 >, \tag{22}$$

i.e., the state is antisymmetric under interchange of the momenta. In particular, the state does not exist if $\vec{k}_2 = \vec{k}_1$. The Pauli principle is *automatically* incorporated in quantum field theory by postulating anti-commutation relations for the creation and annhilation operators. Note also that we now have

$$< \vec{k}_1, \vec{k}_2|\vec{k}_1', \vec{k}_2' >= \frac{1}{2}\left[\delta(\vec{k}_1 - \vec{k}_1')\delta(\vec{k}_2 - \vec{k}_2') - \delta(\vec{k}_1 - \vec{k}_2')\delta(\vec{k}_2 - \vec{k}_1')\right]. \tag{23}$$

Energy-Momentum Operator of Electrodynamics

In quantum electrodynamics we consider electrons, positrons and photon. Ignoring the interactions the appropriate energy-momentum operator P^μ is of the form

$$P^\mu = \int d^3k\hbar k^\mu \sum_r \left[a_r^*(\vec{k})a_r(\vec{k}) + b_r^*(\vec{k})b_r(\vec{k})\right] + \int d^3\omega\hbar\omega^\mu C_\nu^*(\vec{\omega})C^\nu(\vec{\omega}), \quad (24)$$

where

$$k^0 = \sqrt{|\vec{k}|^2 + \left(\frac{mc}{\hbar}\right)^2} \text{ and } \omega^0 = |\vec{\omega}|. \quad (25)$$

Here $a_r^*(\vec{k})$ and $a_r(\vec{k})$ are *electron* creation and annihilation operators, which satisfy

$$\{a_r(\vec{k}), a_s(\vec{k}')\} = 0 \quad (26)$$

and

$$\{a_r(\vec{k}), a_s^*(\vec{k}')\} = \delta_{rs}\delta(\vec{k} - \vec{k}'). \quad (27)$$

Similarly $b_r^*(\vec{k})$ and $b_r(\vec{k})$ are *positron* creation and annihilation operators, which satisfy

$$\{b_r(\vec{k}), b_s(\vec{k}')\} = 0 \quad (28)$$

and

$$\{b_r(\vec{k}), b_s^*(\vec{k}')\} = \delta_{rs}\delta(\vec{k} - \vec{k}'). \quad (29)$$

In both cases the subscripts on the creation and annihilation operators can assume two possible values, corresponding, respectively, to spin "up" and spin "down". Finally, $C_\mu^*(\vec{\omega})$ and $C_\mu(\vec{\omega})$ are *photon* creation and annihilation operators, which satisfy

$$[C_\mu(\vec{\omega}), C_\nu(\vec{\omega})] = 0 \quad (30)$$

and

$$[C_\mu(\vec{\omega}), C_\nu^*(\vec{\omega})] = g_{\mu\nu}\delta(\vec{\omega} - \vec{\omega}'). \quad (31)$$

As the notation seems to imply, the subscripts on the photon creation and annihilation operators can assume four values 0, 1, 2 and 3. One must admit that this is a very peculiar situation, for we know that there are only *two* polarization states of a photon. Furthermore, since $g_{00} = -1$, the C_0 and C_0^* operators satisfy an unusual commutation relation. These are subtle points which deserve careful study, but not in an undergraduate course. Suffice it to say that the apparent difficulties may be circumvented, so they need not concern us here.

The vacuum state $|\Omega>$ of quantum electrodynamics is defined by the conditions

$$a_r(\vec{k})|\Omega> \; = \; 0, \tag{32}$$

$$b_r(\vec{k})|\Omega> \; = \; 0, \tag{33}$$

$$C_\mu(\vec{\omega})|\Omega> \; = \; 0, \tag{34}$$

for all $\vec{k}$ and r and $\vec{\omega}$ and μ. The excited states are formed by applying creation operators for electrons, positrons and photons to the vacuum state vector $|\Omega>$.

It is possible to put the theory of noninteracting particles in a completely satisfactory mathematical form. This situation contrasts greatly with the theory of interactions, which has to this day (1973) not been put into a satisfactory form. In spite of this, there has developed a standard approach to quantum electrodynamics which has afforded us a quantitatively correct understanding of quite bizarre phenomena.

2 The S-Matrix

The S-operator tells one how the interaction picture state vector evolves from $t = -\infty$ to $t = +\infty$; i.e.,

$$|\psi_I(+\infty)> = S|\psi_I(-\infty)> . \tag{35}$$

If the perturbing potential in the interaction picture is denoted by $V_I(t)$, then one finds by time-dependent perturbation theory that

$$S = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar}\right)^n \int_{-\infty}^{\infty} dt_n \ldots \int_{-\infty}^{\infty} dt_1 T\{V_I(t_n)\ldots V_I(t_1)\}, \tag{36}$$

where T is an operator which reorders the V-factors so that the earlier times appear to the right and the later times to the left.

Our objective is to identify the operator $V_I(t)$, and hence the operator S, in terms of creation and annihilation operators. One must take advantage of Lorentz covariance to guide one in the identification of $V_I(t)$. With this objective in mind, let us review the *classical* theory of fields.

Lagrangian Density of the Free Fields

The classical field equations are derivable from the Lagrangian density

$$\mathcal{L}_0 = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \hbar c\bar{\psi}\left(\gamma^\mu\partial_\mu + \frac{mc}{\hbar}\right)\psi, \tag{37}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor. The Lagrangian density $\mathcal{L}_0$ is regarded as a function of ψ, $\bar{\psi}$ and A_μ, plus the spatial and temporal derivatives of these fields.

When a Lagrangian density is specified in terms of a field φ, its derivatives $\partial_\mu\varphi$ and possibly the 4-vector field x too, the variational principle yields

$$\begin{aligned}\delta\int d^4x\mathcal{L}_0(\varphi,\partial_\mu\varphi,x) &= \int d^4\delta\mathcal{L}_0(\varphi,\partial_\mu\varphi,x)\\ &= \int d^4x\left\{\frac{\partial\mathcal{L}_0}{\partial\varphi}\delta\varphi + \delta\partial_\mu\varphi\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\varphi)}\right\}\\ &= \int d^4x\delta\varphi\left\{\frac{\partial\mathcal{L}_0}{\partial\varphi} - \partial_\mu\left[\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\varphi)}\right]\right\} = 0,\end{aligned} \tag{38}$$

providing $\delta\varphi(x) = 0$ and infinity. Treating $\delta\varphi(x)$ as an arbitrary variation elsewhere in spacetime, we conclude that

$$\frac{\partial\mathcal{L}_0}{\partial\varphi} - \partial_\mu\left[\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\varphi)}\right] = 0. \tag{39}$$

This is the Euler-Lagrange field equation for the field $\varphi(x)$.

Since

$$\frac{\partial\mathcal{L}_0}{\partial\bar{\psi}} = -\hbar c\left(\gamma^\mu\partial_\mu + \frac{mc}{\hbar}\right)\psi \text{ and } \frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\bar{\psi})} = 0, \tag{40}$$

it is clear that one of the Euler-Lagrange field equations is simply the Dirac equation

$$\left(\gamma^\mu\partial_\mu + \frac{mc}{\hbar}\right)\psi = 0. \tag{41}$$

The Euler-Lagrange equation that corresponds to variation with respect to ψ is an equation

$$\partial_\mu\bar{\psi}\gamma^\mu - \frac{mc}{\hbar}\bar{\psi} = 0, \tag{42}$$

which is completely equivalent to the Dirac equation.

Turning our attention to the vector potential, we see that

$$\frac{\partial \mathcal{L}_0}{\partial A_\nu} = 0 \text{ and } \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}, \tag{43}$$

and, therefore, the Euler-Lagrange equation is

$$\partial_\mu F^{\mu\nu} = 0, \tag{44}$$

which is Maxwell's field equation in the absence of charges and currents. Thus, the proposed Lagrangian density $\mathcal{L}_0$ does yield the appropriate field equations for the non-interacting fields.

Energy Density of the Noninteracting Fields

Our Lagrangian density can be written in the form

$$\mathcal{L}_0 = \frac{1}{2}(E^2 - H^2) - \hbar c\,\bar{\psi}\left(\gamma^\mu \partial_\mu + \frac{mc}{\hbar}\right)\psi, \tag{45}$$

where $\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{H} = \vec{\nabla} \times \vec{A}$. Notice that $\dot{\vec{A}}$ occurs only in $\vec{E}$ and $\dot{\varphi}$ does not appear at all. Hence the Hamiltonian density is

$$\mathcal{H}_0 = \dot{\vec{A}} \cdot \frac{\partial \mathcal{L}_0}{\partial \dot{\vec{A}}} + \frac{\partial \mathcal{L}_0}{\partial \dot{\psi}}\dot{\psi} - \mathcal{L}_0, \tag{46}$$

where $\frac{\partial \mathcal{L}_0}{\partial \dot{\vec{A}}} = -\vec{E}$ and $\frac{\partial \mathcal{L}_0}{\partial \dot{\psi}} = -\hbar\,\bar{\psi}\gamma^0$. Thus,

$$\begin{aligned}
\mathcal{H}_0 &= (\vec{E} + \vec{\nabla}\varphi) \cdot \vec{E} - \hbar c\,\bar{\psi}\gamma^0 \partial_0 \psi - \mathcal{L}_0 \\
&= \frac{1}{2}(E^2 + H^2) + \vec{\nabla}\varphi \cdot \vec{E} + \hbar c\,\bar{\psi}\left(\vec{\gamma} \cdot \vec{\nabla} + \frac{mc}{\hbar}\right)\psi \\
&= \frac{1}{2}(E^2 + H^2) + \hbar c\,\bar{\psi}\left(\vec{\gamma} \cdot \vec{\nabla} + \frac{mc}{\hbar}\right)\psi - \varphi\vec{\nabla} \cdot \vec{E} + \vec{\nabla} \cdot (\varphi\vec{E}).
\end{aligned} \tag{47}$$

The third term vanishes when the filed equations are satisfied, and the fourth term is a divergence, which contributes nothing to the integrated total energy. Thus, the total field energy can be written

$$H_0 = \int d^3x \left\{\frac{1}{2}(E^2 + H^2) + \hbar c\,\bar{\psi}\left(\vec{\gamma} \cdot \vec{\nabla} + \frac{mc}{\hbar}\right)\psi\right\}. \tag{48}$$

This result can be readily understood if we write it in the form

$$H_0 = \int d^3x \left\{ \frac{1}{2}(E^2 + H^2) + \psi^* \left[c\vec{\alpha} \cdot \left(\frac{\hbar}{i} \vec{\nabla} \right) + mc^2 \beta \right] \psi \right\}, \qquad (49)$$

for the energy density of an electromagnetic field is known to be $\frac{1}{2}(E^2 + H^2)$ in *rationalized* Heaviside-Lorentz units, while the differential operator $c\vec{\alpha} \cdot \left(\frac{\hbar}{i} \vec{\nabla} \right) + mc^2 \beta$ is recognizable as the Hamiltonian of Dirac's single particle theory.

Relationship Between H_0 and P^0 in Quantum Electrodynamics

For consistency one should have $H_0 = cP^0$. H_0 is bilinear in electromagnetic and spinor fields, while P^0 is bilinear in creation and annihilation operators. If one writes

$$\psi(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{\hbar k^0 / mc}} \sum_r \left[a_r(\vec{k}) u_r(\vec{k}) e^{ik \cdot x} + b_r^*(\vec{k}) v_r(\vec{k}) e^{-ik \cdot x} \right] \qquad (50)$$

and

$$A_\mu(x) = (2\pi)^{-3/2} \int \frac{d^3\omega}{\sqrt{2\omega^0 / \hbar c}} \left[C_\mu(\vec{\omega}) e^{i\omega \cdot x} + C_\mu^*(\vec{\omega}) e^{-i\omega \cdot x} \right], \qquad (51)$$

then the expressions for H_0 and cP^0 can be shown to be essentially identical. [Exercise: Substitute ψ and A_μ into H_0 and cast the result into a form like cP^0.]

The one difference between H_0 and cP^0 is that in the latter the creation operators appear to the left and the annihilation operators appear to the right, while this ordering is not observed in H_0. To get perfect agreement one must reorder the factors in H_0. This is accomplished formally by introducing the "normal product", denoted by colons. Thus, we write

$$H_0 = \int d^3x \left\{ \frac{1}{2} : (E^2 + H^2) : + \hbar c \ : \bar{\psi} \left(\vec{\gamma} \cdot \vec{\nabla} + \frac{mc}{\hbar} \right) \psi : \right\}. \qquad (52)$$

Before you get too upset with this, please remember that our objective is simply to guess the appropriate expression for $V_I(t)$ in terms of creation and annihilation operators.

The Interaction

Returning to the Lagrangian density $\mathcal{L}_0$, let us introduce the normal product notation and write

$$\mathcal{L}_0 = \frac{1}{2} : (E^2 + H^2) : -\hbar c \ : \bar{\psi} \left(\gamma^\mu \partial_\mu + \frac{mc}{\hbar} \right) \psi : . \tag{53}$$

Under the usual substitution $\partial_\mu \to \partial_\mu - \frac{iq}{\hbar c} A_\mu$, this yields the Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 + iq \ : \bar{\psi} \gamma^\mu A_\mu \psi := \mathcal{L}_0 + \frac{1}{c} j^\mu A_\mu, \tag{54}$$

where the current density 4-vector is

$$j^\mu = iqc \ \bar{\psi} \gamma^\mu \psi : . \tag{55}$$

If one reviews the derivation of the field equations from the Euler-Lagrange equations, one finds that the additional term in $\mathcal{L}$ produces the following modified equations:

$$\left(\gamma^\mu \partial_\mu + \frac{mc}{\hbar} \right) \psi - \frac{iq}{\hbar c} \gamma^\mu A_\mu \psi = 0 \tag{56}$$

$$\partial_\nu F^{\mu\nu} = \frac{1}{c} j^\mu. \tag{57}$$

These are recognized as the Dirac and Maxwell field equations in the presence of an interaction.

The interaction term in the Lagrangian leads to a modification of the energy operator. Clearly, one gets now

$$H = \int d^3x \left\{ \frac{1}{2} : (E^2 + H^2) : +\hbar c \ : \bar{\psi} \left(\vec{\gamma} \cdot \vec{\nabla} + \frac{mc}{\hbar} \right) \psi : -iq \ : \bar{\psi} \gamma^\mu A_\mu \psi : \right\} . \tag{58}$$

As a result, we shall identify the interaction Hamiltonian as

$$V_I(t) = \int d^3x [-iq \ : \bar{\psi} \gamma^\mu A_\mu \psi :] = -\frac{1}{c} \int d^3x j^\mu A_\mu, \tag{59}$$

and the S-operator is consequently

$$S = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{\hbar c} \right)^n \int d^4x_n \ldots \int d^4x_1 T\{ j^{\mu_n}(x_n) A_{\mu_n}(x_n) \ldots j^{\mu_1}(x_1) A_{\mu_1}(x_1) \}. \tag{60}$$

This can be written in terms of creation and annihilation operators by substituting our expansions of $\psi(x)$, $\bar{\psi}(x)$ and $A_\mu(x)$. We shall not actually do this, however, but instead we shall turn to the question of using "Feynman diagrams" to evaluate S-matrix elements.

3 Origin of Feynman Diagrams and Rules

In general, we shall be interested in evaluating matrix elements of the S-operator (60) between eigenstates of P^μ in which there are definite particles present. For example, the scattering of an electron of momentum $\hbar\vec{k}$ and a photon of momentum $\hbar\vec{\omega}$ to yield an electron of momentum $\hbar\vec{k}'$ and a photon of momentum $\hbar\vec{\omega}'$ would be described by the S-matrix element

$$< \vec{k}, s'; \vec{\omega}', \hat{e}'|S|\vec{k}, s; \vec{\omega}, \hat{e} >,$$

where s and s' are electron spin labels and $\hat{e}$ and $\hat{e}'$ are photon polarization vectors. In particular, the initial state is given by

$$|\vec{k}, s; \vec{\omega}, \hat{e} >= a_s^*(\vec{k}) e^\mu C_\mu^*(\vec{\omega})|\Omega >, \tag{61}$$

where $|\Omega >$ is the vacuum state. In the case of positron-photon scattering the corresponding initial state would be

$$|\vec{k}, s; \vec{\omega}, \hat{e} >= b_s^*(\vec{k}) e^\mu C_\mu^*(\vec{\omega})|\Omega > . \tag{62}$$

Whenever confusion may arise we shall distinguish electron and positron states by appending a $-$ or a $+$ to the state vector label.

The operator $A_\mu(x)$ can either create or annihilate a photon. Therefore, if $\vec{\omega}' \neq \vec{\omega}$ or $\hat{e}' \neq \hat{e}$, it will be necessary to have *at least two* $A_\mu(x)$ factors. Only terms in the S-operator expansion with n even, and at least equal to 2, contribute to electron-photon scattering. Any extra $A_\mu(x)$ factors must be paired up so that whatever one creates, another annihilates.

One $\psi(x)$ factor is required to annihilate the initial electron, and one $\bar{\psi}(x)$ factor is required to create the final electron. Since $n \geq 2$, there is at least one extra $\psi(x)$ and at least one extra $\bar{\psi}(x)$ factor, which must annihilate each other's creation.

It should be plausible that the evaluation of S-matrix elements will involve the consideration of factors such as the following:

$$< \Omega|A_\mu(x)|\vec{\omega}, \hat{e} >, \qquad < \vec{\omega}, \hat{e}|A_\mu(x)|\Omega >,$$

$$< \Omega|\psi(x)|\vec{k},s,- >, \qquad < \vec{k},s,-|\bar{\psi}(x)|\Omega >,$$
$$< \Omega|\bar{\psi}(x)|\vec{k},s,+ >, \qquad < \vec{k},s,+|\psi(x)|\Omega >,$$
$$< \Omega|T(A_\mu(x)A_\nu(x'))|\Omega >, \quad < \Omega|T(\psi(x)\bar{\psi}(x')|\Omega > .$$

Therefore, we shall begin by evaluating each of these basic factors, and identifying each with a particular element of a "Feynman diagram."

Factors associated with External Lines

Photon Lines

By Eq. (51) we obviously have

$$\begin{aligned}
< \Omega|A_\mu(x)|\vec{\omega},\hat{e} > &= (2\pi)^{-3/2}\int \frac{d^3\omega'}{\sqrt{2\omega'^0/\hbar c}} < \Omega|C_\mu(\vec{\omega})|\vec{\omega},\hat{e} > e^{i\omega'\cdot x} \\
&= (2\pi)^{-3/2}\int \frac{d^3\omega'}{\sqrt{2\omega'^0/\hbar c}} < \Omega|C_\mu(\vec{\omega})e^\nu C^*_\nu(\vec{\omega})|\Omega > e^{i\omega'\cdot x} \\
&= (2\pi)^{-3/2}\int \frac{d^3\omega'}{\sqrt{2\omega'^0/\hbar c}} e^\nu < \Omega|[C_\mu(\vec{\omega}), C^*_\nu(\vec{\omega})]|\Omega > e^{i\omega'\cdot x} \\
&= (2\pi)^{-3/2}\int \frac{d^3\omega'}{\sqrt{2\omega'^0/\hbar c}} e^\nu g_{\mu\nu}\delta(\vec{\omega}'-\vec{\omega})e^{i\omega'\cdot x} \\
&= (2\pi)^{-3/2}\frac{e_\mu}{\sqrt{2\omega^0/\hbar c}}e^{i\omega\cdot x}, \qquad (63)
\end{aligned}$$

where $\omega^0 = |\vec{\omega}|$. Similarly, we find

$$< \vec{\omega},\hat{e}|A_\mu(x)|\Omega >= (2\pi)^{-3/2}\frac{e_\mu}{\sqrt{2\omega^0/\hbar c}}e^{-i\omega\cdot x}. \qquad (64)$$

The factor $(2\pi)^{-3/2}e_\mu/\sqrt{2\omega^0/\hbar c}$ will be associated in a Feynman diagram with an *external photon line*, portrayed by a wiggly line which leaves the diagram.

Electron-Positron Lines

By Eq. (50) we have

$$< \Omega|\psi(x)|\vec{k},s,- >= (2\pi)^{-3/2}\frac{u_s(\vec{k})}{\sqrt{\hbar k^0/mc}}e^{ik\cdot x} \qquad (65)$$

and

$$< \vec{k}, s, +|\psi(x)|\Omega >= (2\pi)^{-3/2} \frac{v_s(\vec{k})}{\sqrt{\hbar k^0/mc}} e^{-ik\cdot x}, \tag{66}$$

where

$$k^0 = +\sqrt{|\vec{k}|^2 + \left(\frac{mc}{\hbar}\right)^2}. \tag{67}$$

Similarly, since

$$\bar{\psi}(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{\hbar k^0/mc}} \sum_s \left[a_s^*(\vec{k})\bar{u}_s(\vec{k})e^{-ik\cdot x} + b_s(\vec{k})\bar{v}_s(\vec{k})e^{ik\cdot x}\right], \tag{68}$$

we have

$$< \Omega|\bar{\psi}(x)|\vec{k}, s, + >= (2\pi)^{-3/2} \frac{\bar{v}_s(\vec{k})}{\sqrt{\hbar k^0/mc}} e^{ik\cdot x} \tag{69}$$

and

$$< \vec{k}, s, -|\bar{\psi}(x)|\Omega >= (2\pi)^{-3/2} \frac{\bar{u}_s(\vec{k})}{\sqrt{\hbar k^0/mc}} e^{-ik\cdot x}. \tag{70}$$

With each *incoming electron line* we associate a factor

$$(2\pi)^{-3/2} u_s(\vec{k})/\sqrt{\hbar k^0/mc},$$

while with each *outgoing electron line* we associate a factor

$$(2\pi)^{-3/2} \bar{u}_s(\vec{k})/\sqrt{\hbar k^0/mc},$$

portraying such lines with solid straight arrows directed toward the future.

With each *incoming positron line* we associate a factor

$$(2\pi)^{-3/2} \bar{v}_s(\vec{k})/\sqrt{\hbar k^0/mc},$$

while with each *outgoing positron line* we associate a factor

$$(2\pi)^{-3/2} v_s(\vec{k})/\sqrt{\hbar k^0/mc},$$

portraying such lines with solid straight arrows directed toward the past.

Factors associated with Internal Lines

Photon Lines

The *internal photon line* is associated with the matrix element

$$< \Omega|T(A_\mu(x)A_\nu(x'))|\Omega >,$$

where a photon is created only to be destroyed subsequently. Let us consider the evaluation of this vacuum expectation value for $x^0 > x'^0$. In this case, we have

$$\begin{aligned}
&< \Omega|T(A_\mu(x)A_\nu(x'))|\Omega >=< \Omega|A_\mu(x)A_\nu(x')|\Omega > \\
&= (2\pi)^{-3/2}\int \frac{d^3\omega}{\sqrt{2\omega^0/\hbar c}}\int \frac{d^3\omega'}{\sqrt{2\omega'^0/\hbar c}} < \Omega|C_\mu(\vec{\omega})C^*_\nu(\vec{\omega}')|\Omega > e^{i\omega\cdot x}e^{-i\omega'\cdot x'} \\
&= (2\pi)^{-3/2}\int \frac{d^3\omega}{\sqrt{2\omega^0/\hbar c}}\int \frac{d^3\omega'}{\sqrt{2\omega'^0/\hbar c}}\delta(\vec{\omega}-\vec{\omega}')e^{i\omega\cdot x}e^{-i\omega'\cdot x'} \\
&= (2\pi)^{-3/2}g_{\mu\nu}\hbar c\int \frac{d^3\omega}{2\omega^0}e^{i\omega\cdot(x-x')} \\
&= (2\pi)^{-3/2}g_{\mu\nu}\hbar c\int \frac{d^3\omega}{2|\omega|}e^{i[\vec{\omega}\cdot(\vec{x}-\vec{x}')-|\vec{\omega}|(x^0-x'^0)]} \\
&= (2\pi)^{-3/2}g_{\mu\nu}\hbar c\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int \frac{d^4\omega}{\omega^2-i\epsilon}e^{i\omega\cdot(x-x')}, \qquad (71)
\end{aligned}$$

where $\omega^2 := |\vec{\omega}|^2 - (\omega^0)^2$. The last expression involves a contour integration over ω^0 from $-\infty$ to $+\infty$, with a pole just *below* the real axis at $\omega^0 = +|\vec{\omega}|$ and a pole just *above* the real axis at $\omega^0 = -|\vec{\omega}|$. Since $x^0 > x'^0$ was assumed, the contour may be closed using an infinitely large semicircle in the lower half complex plane, for the factor $e^{-i\omega^0(x^0-x'^0)}$ vanishes there. Thus, one just gets the residue at the pole near $\omega^0 = +|\vec{\omega}|$. After the integral is evaluated, you let the poles approach the real axis, where they insure the energy-momentum relation.

If one considers the case $x^0 < x'^0$, one finds that the *same* final formula applies. Thus, with each internal photon line we shall associate a factor

$$\frac{-i\hbar c g_{\mu\nu}}{(2\pi)^4(\omega^2-i\epsilon)},$$

with the understanding that one must eventually perform an integration over $d^4\omega$ and let $\epsilon \to 0$. [Note that contour integrals almost identical to that which we introduced here are employed in studies of the classical electromagnetic wave equation. See Jackson, *Classical Electrodynamics*.]

Electron-Positron Lines

Internal electron-positron lines are denoted by solid arrows terminating within the Feynman diagram. With each such line we associate the matrix element

$$< \Omega|T(\psi(x)\bar{\psi}(x')|\Omega > .$$

If $x^0 > x'^0$, this gives

$$\begin{aligned} &< \Omega|T(\psi(x)\bar{\psi}(x'))|\Omega > = < \Omega|\psi(x)\bar{\psi}(x')|\Omega > \\ &= (2\pi)^{-3}\sum_s \int \frac{d^3k}{\sqrt{\hbar k^0/mc}} \sum_{s'} \int \frac{d^3k'}{\sqrt{\hbar k'^0/mc}} < \Omega|a_s(\vec{k})a^*_{s'}(\vec{k}')|\Omega > \\ &\qquad u_s(\vec{k})\bar{u}_{s'}(\vec{k}')e^{ik\cdot x}e^{-ik'\cdot x'} \\ &= (2\pi)^{-3}\sum_s \int \frac{d^3k}{\sqrt{\hbar k^0/mc}} \sum_{s'} \int \frac{d^3k'}{\sqrt{\hbar k'^0/mc}} \delta_{ss'}\delta(\vec{k}-\vec{k}') \\ &\qquad u_s(\vec{k})\bar{u}_{s'}(\vec{k}')e^{ik\cdot x}e^{-ik'\cdot x'} \\ &= (2\pi)^{-3}\int \frac{d^3k}{[\hbar k^0/mc]} \sum_s u_s(\vec{k})\bar{u}_s(\vec{k})e^{ik\cdot(x-x')}. \end{aligned} \tag{72}$$

However, one may show that

$$\sum_s u_s(\vec{k})\bar{u}_s(\vec{k}) = \frac{-i\gamma^\mu k_\mu + \frac{mc}{\hbar}}{2(mc/\hbar)}. \tag{73}$$

We now have (for $x^0 > x'^0$)

$$\begin{aligned} < \Omega|T(\psi(x)\bar{\psi}(x'))|\Omega > &= (2\pi)^{-3}\int \frac{d^3k}{2k^0}\left(-i\gamma^\mu k_\mu + \frac{mc}{\hbar}\right)e^{ik\cdot(x-x')} \qquad (74) \\ &= (2\pi)^{-3}\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int d^4k \frac{-i\gamma^\mu k_\mu + \frac{mc}{\hbar}}{k^2+(mc/\hbar)^2 - i\epsilon}e^{ik\cdot(x-x')}, \end{aligned}$$

and one can show that the same final formula holds if $x^0 < x'^0$. Hence, with each internal electron-positron line we shall associate a factor

$$-i\frac{-i\gamma^\mu k_\mu + mc/\hbar}{k^2+(mc/\hbar)^2 - i\epsilon},$$

with the understanding that eventually one must integrate over d^4k and take the limit $\epsilon \to 0$.

Factors associated with Vertices

At each vertex precisely one arrow enters and one arrow leaves. In addition, one photon line emends at the vertex. The integration over d^4x associatd with each vertex always yields $(2\pi)^4c^{-1}$ times a δ-function which conserves both momentum and energy at the vertex. For example, if an electron has 4-momentum k initially and 4-momentum k' after emitting a photon of 4-momentum ω', then the integration over d^4x yields

$$(2\pi)^4c^{-1}\delta^4(k-k'-\omega').$$

The factor c^{-1} comes from the fact that d^4x involves dt rather than $dx^0 = cdt$.

In addition, each vertex brings in a factor $(i/\hbar c)$ and a factor $i(-e)c$. Thus, we shall associate with each vertex a factor

$$(2\pi)^4\frac{e}{\hbar c}\delta^4,$$

where the argument of the δ-function is a 4-momentum to be conserved at the vertex.

At the beginning of the next section we shall summarize the Feynman rules, and apply them to several simple calculations to illustrate their practical utility.

4 Evaluation of S-Matrix Elements

In order to evaluate a scattering cross section or a transition rate it is necessary to calculate an S-matrix element. The evaluation of the latter is greatly facilitated by the use of Feynman diagrams. In the previous chapter we have given some indication of the origin of the Feynman diagram technique, but we were forced to sweep all subtleties under the rug. If later you should desire to rectify the shortcomings of our rather superficial approach, I suggest that you consult not only a standard textbook on quantum electrodynamics, but also some of the original research papers, where the footnotes often provide considerable insight into the process of building such a theory.

At the end of the second world war, Feynman, Schwinger and Tomonaga independently developed three different versions of relativistic quantum electrodynamics. The equivalence of the three approaches was subsequently established by F. J. Dyson, whose papers I recommend.

Dyson, Physical Review **75**, 486 (1949); **75**, 1736 (1949).

Thumb through these papers to spot points of contact with my notes before taking on the formidable task of reading them.

The basic theorem relating time-ordered and normal-products of field operators, which we found it necessary to gloss over, is discussed in a paper of Wick.

Wick, Physical Review **80**, 268 (1950).

We have indicated that there are certain subtle problems connected with photon polarization. These prob lems are treated by Gupta and Bleuler.

Gupta, Proceedings of the Physical Society (London) **63**A, 681 (1950); Bleuler, Helvetica Physica Acta **23**, 567 (1950).

While we are enumerating useful references, let me also call attention to an important article by Møller concerning the relation between the S-matrix element and the cross section.

Møller, Det Kgl. Danske Videnskabernes Selskab Matematisk-fysiske Meddelelser **23**, 1 (1945).

Do not panic! The article is in English, not Danish. The subject is difficult, however.

Summary of Feynman Diagram Factors

1. External electron-positron lines:

	electron	positron
incoming	$(2\pi)^{-3/2}\frac{u_s(\vec{k})}{\sqrt{\hbar k^0/mc}}$	$(2\pi)^{-3/2}\frac{\bar{v}_s(\vec{k})}{\sqrt{\hbar k^0/mc}}$
outgoing	$(2\pi)^{-3/2}\frac{\bar{u}_s(\vec{k})}{\sqrt{\hbar k^0/mc}}$	$(2\pi)^{-3/2}\frac{v_s(\vec{k})}{\sqrt{\hbar k^0/mc}}$

2. Internal electron-positron lines:

$$-i(2\pi)^{-4}\frac{-i\gamma^\mu k_\mu + \left(\frac{mc}{\hbar}\right)}{k^2 + \left(\frac{mc}{\hbar}\right)^2 - i\epsilon}$$

Here ϵ is a small positive number which eventually goes to zero.

3. Vertices:
$$(2\pi)^4 \frac{e}{\hbar c} \delta^4 \gamma^\mu$$
Here the delta function argument vanishes when energy and momentum are conserved at the vertex.

4. External photon lines:
$$(2\pi)^{-3/2} \frac{e_\mu}{\sqrt{2\omega^0/\hbar c}}$$
Here e^μ is the polarization 4-vector.

5. Internal photon lines:
$$-i(2\pi)^4 \hbar c \frac{g_{\mu\nu}}{\omega^2 - i\epsilon}$$
Here ϵ is a small positive number which eventually goes to zero, and $g_{\mu\nu}$ is the metric tensor.

Step by Step Instructions

In order to avoid errors I suggest following these rules when you attempt to evaluate any S-matrix element.

Step 1: To each electron and positron in the initial and final states assign a distinct momentum and a distinct spin index. To each photon in the initial and final states assign a distinct momentum and a distinct polarization vector.

Step 2: Choose a value for the integer n, which is the number of vertices in the Feynman diagrams to be constructed. Draw all possible topologically distinct diagrams with n vertices, having the proper initial particles and proper final particles, labelled in accord with spep one. Each vertex should have precisely one photon line, on incoming arrow and one outgoing arrow.

Step 3: To each vertex assign a distinct spacetime index (such as μ), and to each internal line assign a distinct symbol for the 4-momentum and a distinct symbol for a small positive quantity (such as ϵ).

Step 4: Proceed along one complete electron-positron line, and write factors from right to left in accordance with the rules on the previous page. Follow the direction of the arrows, and remember to use the proper symbols introduced in steps one through three. Repeat this same procedure for each complete electron-positron line in the diagram, including any closed loops.

Step 5: Write additional factors for each internal and each external photon line in the diagram, remembering to use the proper symbols introduced in steps one through three. The order of these factors is immaterial.

Step 6: Integrate now over each 4-momentum associated with an internal line. Then let each one of the small positive quantities associated with internal lines tend to zero. The result is the S-matrix element corresponding to the diagram considered.

Example One (Compton Scattering)

Let us denote the initial electron momentum variable and spin index by $\vec{k}, s$ and let us denote the final electron momentum variable and spin index by $\vec{k}', s'$. Let the photon in the initial state have momentum variable and polarization $\vec{\omega}, \hat{e}$, while the corresponding quantities for the photon in the final state are $\vec{\omega}', \hat{e}'$.

For $n = 2$ the only topologically distinct Feynman diagrams are the following:

FIGURE 1 HERE

The various lines have been labelled in accord with steps one through three.

Let us follow steps four through six in order to write the S-matrix element corresponding to the diagram on the left. We begin with the single complete electron line, proceeding in the direction of the arrows. At the end of step four we have accumulated the following factors, which we display from leftmost to rightmost:

1. $(2\pi)^{-3/2} \frac{\bar{u}_{s'}(\vec{k}')}{\sqrt{\hbar k'^0/mc}}$
2. $(2\pi)^4 \frac{e}{\hbar c} \delta^4(k' + \omega' - q)\gamma^\nu$
3. $-i(2\pi)^{-4} \frac{-i\gamma^\alpha q_\alpha + \frac{mc}{\hbar}}{q^2 + \left(\frac{mc}{\hbar}\right)^2 - i\epsilon}$

4. $(2\pi)^4 \frac{e}{\hbar c} \delta^4(q-k-\omega)\gamma^\mu$

5. $(2\pi)^{-3/2} \frac{u_s(\vec{k})}{\sqrt{\hbar k^0/mc}}$

The product of these factors is given by

$$-2\pi i \left(\frac{e}{\hbar c}\right)^2 \frac{\delta^4(k'+\omega'-q)\delta^4(q-k-\omega)}{\sqrt{\hbar k'^0/mc}\sqrt{\hbar k^0/mc}} \frac{\bar{u}_{s'}(\vec{k}')\gamma^\nu \left[-i\gamma^\alpha q_\alpha + \left(\frac{mc}{\hbar}\right)\right]\gamma^\mu u_s(\vec{k})}{q^2 + \left(\frac{mc}{\hbar}\right)^2 - i\epsilon}.$$

As you rearrange factors be careful not to commute matrices thoughtlessly. At the end of step five we have

$$-(2\pi)^2 i \left(\frac{e}{\hbar c}\right)^2 \frac{\delta^4(k'+\omega'-q)\delta^4(q-k-\omega)}{\sqrt{\hbar k'^0/mc}\sqrt{2\omega'^0/\hbar c}\sqrt{\hbar k^0/mc}\sqrt{2\omega^0/\hbar c}} \frac{\bar{u}_{s'}(\vec{k}')(\gamma^\nu e'_\nu)\left[-i\gamma^\alpha q_\alpha + \left(\frac{mc}{\hbar}\right)\right](\gamma^\mu e_\mu) u_s(\vec{k})}{q^2 + \left(\frac{mc}{\hbar}\right)^2 - i\epsilon}.$$

At the end of step six the integration yields the following expression:

$$-(2\pi)^{-2} i \left(\frac{e}{\hbar c}\right)^2 \frac{\delta^4(k'+\omega'-k-\omega)}{\sqrt{\hbar k'^0/mc}\sqrt{2\omega'^0/\hbar c}\sqrt{\hbar k^0/mc}\sqrt{2\omega^0/\hbar c}} \frac{\bar{u}_{s'}(\vec{k}')(\gamma^\nu e'_\nu)\left[-i\gamma^\alpha (k_\alpha+\omega_\alpha) + \left(\frac{mc}{\hbar}\right)\right](\gamma^\mu e_\mu) u_s(\vec{k})}{(k+\omega)^2 + \left(\frac{mc}{\hbar}\right)^2}$$

This is half the S-matrix. As an exercise, write out the corresponding expression for the second $n = 2$ diagram shown earlier. The sum of the two expressions gives the complete S-matrix element for Compton scattering. Notice the survival of one energy-momentum conserving delta function. The total initial energy-momentum must be the same as the total final energy-momentum.

Example Two (Møller Scattering of Two Electrons)

For $n = 2$ the only topologically distinct Feynman diagrams are the following:

FIGURE 2 HERE

The various lines have been labelled in accord with steps one through three. As an exercise evaluate both contributions to the S-matrix element. The whole S-matrix element should turn out to be consistent with the Pauli principle. This requires changing the sign of the contribution from the second diagram. Whenever two complete electron- positron lines cross, as in diagram two, such sign changes may be necessary.

Example Three (Mott Scattering)

A good deal of interest in Mott scattering has been expressed by students recently. As a result, I have promised to perform a calculation of the Mott scattering cross section. Mott scattering concerns the scattering of an electron in the Coulomb field of a nucleus.

Because a satisfactory field theory of nuclear particles does not exist, we shall be forced to treat the nuclear field in a very crude way. What we shall do is treat the interaction between the electron and the nucleus in terms of a *non-quantized* Coulomb potential.

There is just one $n = 1$ diagram for Mott scattering.

FIGURE 3 HERE

Notice the special X-symbol attached to the "photon" line. This indicates a *classical* interaction via the 4-potential $A_\mu(x)$.

A vertex attached to such a classical photon line is treated differently from the usual vertex encountered in our other calculations. Here the appropriate vertex factor has the form

$$\frac{e}{\hbar}\gamma^\mu \int e^{i(k-k')\cdot x} A_\mu(x) d^4x.$$

The entire S-matrix element assumes the form

$$(2\pi)^{-3}\left(\frac{e}{\hbar}\right)\frac{\bar{u}_{s'}(\vec{k}')\gamma^\mu u_s(\vec{k})}{\sqrt{\hbar k'^0/mc}\sqrt{\hbar k^0/mc}}\int e^{i(k-k')\cdot x} A_\mu(x) d^4x.$$

In the present problem the 4-potential is given by

$$A_0(x) = -\frac{Ze}{4\pi|\vec{x}|}, \quad \vec{A}(x) = 0. \tag{75}$$

Hence, the S-matrix element is given by

$$-(2\pi)^{-2} Z \left(\frac{e^2}{4\pi\hbar c}\right) \frac{\bar{u}_{s'}(\vec{k}')\gamma^0 u_s(\vec{k})}{\sqrt{\hbar k'^0/mc}\sqrt{\hbar k^0/mc}} \delta(k'^0 - k^0) \int \frac{d^3x}{|\vec{x}|} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}. \tag{76}$$

The remaining integral may be evaluated easily in terms of polar coordinates, providing a convergence factor such as $\exp(-\lambda r)$ is introduced. After the integration is performed, you let λ go to zero. Physically this is equivalent to taking account of the screening of the nuclear Coulomb field by the atomic electrons.

The result for the integral is simply $4\pi/|\vec{k}-\vec{k}'|^2$, so the S-matrix element has the value

$$-\frac{Z}{\pi|\vec{k}-\vec{k}'|^2} \left(\frac{e^2}{4\pi\hbar c}\right) \frac{\bar{u}_{s'}(\vec{k}')\gamma^0 u_s(\vec{k})}{\sqrt{\hbar k'^0/mc}\sqrt{\hbar k^0/mc}} \delta(k'^0 - k^0). \tag{77}$$

The quantity in brackets may be recognized as the fine structure constant, with value 1/137. It should also be noted that for non-relativistic electrons this expression reduces to

$$\frac{iZ}{\pi|\vec{k}-\vec{k}'|^2} \left(\frac{e^2}{4\pi\hbar c}\right) \delta_{s's} \delta(k'^0 - k^0). \tag{78}$$

In the next chapter we shall discuss how to get a scattering cross section from an S-matrix element such as this.

5 Transition Rates and Differential Scattering Cross Sections

Earlier in this course we established that the S-matrix element for stimulated emission and absorption of radiation involves an energy-conserving delta function if the harmonic perturbation remains on indefinitely. In the previous chapter we encountered such an energy-conserving delta function again in our expression for the S-matrix element for Mott scattering. In the case of Compton scattering and Møller scattering one finds not only such an energy-conserving delta function but also momentum-conserving delta functions. In no case is it legitimate to square delta functions in evaluating transition probabilities. Thus, the evaluation of transition rates and differential scattering cross sections involves subtleties, to which we now turn our attention.

For definiteness, let us consider the evaluation of the Mott scattering cross section. Here we encounter the mathematically meaningless quantity

$$[\delta(k'^0 - k^0)]^2.$$

It arises because we employed time-dependent perturbation theory for a perturbation which is left on indefinitely. One may imagine, however, replacing the delta function in the S-matrix element by some strongly peaked function, the width of which decreases as the time T during which the perturbation is effective increases. This strongly peaked function may be squared, yielding a function which can be approximated in turn by a delta function. One may establish that this course of action leads in effect to the replacement of the square of the delta function by

$$\delta(k'^0 - k^0)\frac{cT}{2\pi}.$$

Hence, the transition rate for Mott scattering is given by

$$\left(\frac{Ze^2 m}{4\pi^2\hbar^2|\vec{k}-\vec{k}'|^2}\right)^2 \frac{c}{2\pi}\frac{|u^*_{s'}(\vec{k}')u_s(\vec{k})|^2}{(k^0)^2}\delta(k'^0 - k^0), \tag{79}$$

where we have utilized the relations $\gamma^0 = -i\beta$, $\bar{u} = u^*\beta$ and $\beta^2 = 1$.

In an actual scattering experiment one detects any electron which is scattered into a certain solid angle $d\Omega$ subtended by the detector. Thus, in practice, one detects scattered electrons with momenta $\hbar\vec{k}'$ in a certain range. One should, therefore, integrate the transition rate over such a range, taking into account the density of possible momentum states.

Very early in this course we showed how to evaluate the density of states factor by putting the system in a box, on the surface of which periodic boundary conditions are imposed. See our discussion of the black-body radiation laws. Our choice of normalization of the one-electron states corresponds to using a box of volume $(2\pi)^2$. In this case the density of states is given by

$$\rho = \frac{d^3k'}{dk'^0} = \frac{k'^2 dk' d\Omega}{dk'^0}. \tag{80}$$

If one integrates the previously given transition rate with respect to k'^0, using this density of states as a weighting factor, the integrated transition rate is

$$\left(\frac{Ze^2 m}{4\pi^2\hbar^2|\vec{k}-\vec{k}'|^2}\right)^2 \frac{c}{2\pi}|u^*_{s'}(\vec{k}')u_s(\vec{k})|^2_{k'^0=k^0}\left(\frac{k^2 dk}{(k^0)^2 dk^0}\right) d\Omega. \tag{81}$$

However,

$$(k'^0)^2 = (k')^2 + \left(\frac{mc}{\hbar}\right)^2, \tag{82}$$

so

$$\frac{k^2 dk}{(k^0)^2 dk^0} = \frac{k}{k^0} = \frac{v}{c}, \tag{83}$$

where v is the velocity of the incident electron. Therefore, the transition rate assumes the form

$$\left(\frac{Ze^2 m}{2\pi^2\hbar^2|\vec{k}-\vec{k}'|^2}\right)^2 |u^*_{s'}(\vec{k}')u_s(\vec{k})|^2 \frac{v}{(2\pi)^3} d\Omega. \tag{84}$$

Since one has one incident electron in a volume $(2\pi)^3$, the flux of incident electrons is given by

$$\text{flux} = v/(2\pi)^3. \tag{85}$$

Hence, the differential scattering cross section is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2 m}{2\pi^2\hbar^2|\vec{k}-\vec{k}'|^2}\right)^2 |u^*_{s'}(\vec{k}')u_s(\vec{k})|^2. \tag{86}$$

The Mott scattering cross section may be compared to the familiar Rutherford scattering formula for nonrelativistic electrons. Note that

$$|\vec{k}-\vec{k}'|^2 = k^2 + k'^2 - 2\vec{k}\cdot\vec{k}' = 2k^2(1-\cos\theta) = 4k^2\sin^2(\theta/2). \tag{87}$$

Hence

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2 m}{8\pi\hbar^2 k^2}\right)^2 \frac{|u^*_{s'}(\vec{k}')u_s(\vec{k})|^2}{\sin^4(\theta/2)}. \tag{88}$$

In comparing this with the standard Rutherford formula you should not forget that we are using rationalized Heaviside-Lorentz units,

$$(e^2/4\pi)_{HL} = (e^2)_{Gaussian} = (e^2/4\pi\epsilon_0)_{MKS}. \tag{89}$$

In our units the traditional Rutherford formula is simply

$$\left(\frac{d\sigma}{d\Omega}\right)_{Rutherford} = \left(\frac{Ze^2 m}{8\pi\hbar^2 k^2}\right)^2 \frac{\delta_{s's}}{\sin^4(\theta/2)}. \tag{90}$$

Hence, the relativistic Mott scattering formula may be obtained from the Rutherford nonrelativistic formula by replacing the Kronecker delta by the factor

$$|u^*_{s'}(\vec{k}')u_s(\vec{k})|^2,$$

which turns out to have the value

$$|u^*_{s'}(\vec{k}')u_s(\vec{k})|^2 = \delta_{s's}\left[1 + \frac{1}{2}\left(\frac{\hbar k^0}{mc} - 1\right)(1+\cos\theta)\right]^2 + \frac{1}{4}\left(\frac{\hbar k^0}{mc} - 1\right)^2 |(\vec{\sigma}\cdot\hat{n})_{s's}|^2 \sin^2\theta. \tag{91}$$

when $k'^0 = k^0$. Hence, if one deals with polarized electrons, i.e., electrons whose spins are known, then the scattering cross section will be different for scattering events in the plane perpendicular to the quantization axis and for scattering events in any plane parallel to the quantization axis. For the first case

$$|(\vec{\sigma}\cdot\hat{n})_{s's}|^2 = \delta_{s's}, \tag{92}$$

while for the second case

$$|(\vec{\sigma}\cdot\hat{n})_{s's}|^2 = \delta_{-s's}. \tag{93}$$

Most often, however, one employs an unpolarized electron beam, and one detects all the scattered electrons without regard to their spin states. In this case, one should average over s and sum over the two possible values of s'. One thus obtains

$$\frac{1}{2}\sum_s\sum_{s'}|u^*_{s'}(\vec{k}')u_s(\vec{k})|^2 = 1 + \left(\frac{\hbar k}{mc}\right)^2 \cos^2(\theta/2). \tag{94}$$

For unpolarized electrons, the Mott scattering cross section therefore assumes the form

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2 m}{8\pi\hbar^2 k^2}\right)^2 \frac{1 + \left(\frac{\hbar k}{mc}\right)^2 \cos^2(\theta/2)}{\sin^4(\theta/2)}. \tag{95}$$

Standard textbooks may be consulted for more sophisticated methods of handling polarization questions. I suggest the following as a useful acquisition:

Bjorken and Drell, *Relativistic Quantum Mechanics* (McGraw-Hill).