Chapter 12

Voyage into the Past

We saw earlier that even in the absence of gravitational fields (spacetime curvature) it is possible, by traveling at near the speed of light, to travel into the earth's future at a more rapid rate than the passengers of "spaceship earth." In the presence of a gravitational field even clocks at rest tick at different rates. Thus clocks on the surface of a white dwarf star tick more slowly than clocks far from the surface, as is seen from the red shift in the absorption lines of light emitted from such compact stars. This provides yet another mechanism that can be used to travel into the future. But what about travel into the past?[1]

The singularity of the Schwarzschild metric and the ring singularity of the Kerr metric (for $|a| < M$) are both hidden behind event horizons, across which a voyager may pass and so reach the singularity, concerning which experience he cannot communicate with anyone who remains outside the black hole. However, when $|a| > M$, the ring singularity of the Kerr metric is "naked." One can approach it as closely as one dares, and return to talk about the experience. We shall see that access to the ring makes possible, at least in principle, travel into one's own past.

Free fall toward a hyperextreme Kerr object along its symmetry axis

On the axis of a Kerr object the Lagrangian reduces to

$$L = \left(1 - \frac{(2GM/c^2)r}{r^2 + a^2}\right)\dot{t}^2 - \frac{1}{c^2}\left(1 - \frac{(2GM/c^2)r}{r^2 + a^2}\right)^{-1}\dot{r}^2 \ ,$$

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[1] Such a possibility is precluded if you insist that the theory be *causal*, i.e., that it always be possible to formulate an initial value problem with a unique solution.

where dots denote derivatives with respect to proper time τ. Because the Lagrangian is independent of t, the energy

$$E := mc^2 \left(1 - \frac{(2GM/c^2)r}{r^2 + a^2}\right) \dot{t}$$

of a freely falling object is conserved. The other relevant equation of motion can be obtained most easily by setting the Lagrangian equal to unity.[2] In this way we obtain

$$\left(\frac{\dot{r}}{c}\right)^2 - \frac{(2GM/c^2)r}{r^2 + a^2} = \left(\frac{E}{mc^2}\right)^2 - 1 = \text{const.} \tag{12.1}$$

Differentiating this equation with respect to τ, we further obtain the equation

$$\ddot{r} + \frac{GM(r^2 - a^2)}{(r^2 + a^2)^2} = 0, \tag{12.2}$$

which shows that $\ddot{r} = 0$ when $r = \pm a$.

Equation (12.1) can be expressed in the suggestive form

$$\frac{1}{2} m\dot{r}^2 + V_{\text{eff}} = E_{\text{class}} ,$$

where

$$E_{\text{class}} := \frac{1}{2} mc^2 [(\frac{E}{mc^2})^2 - 1]$$

plays the role of the classical total energy, and where the *effective potential*

$$V_{\text{eff}} := -\frac{GMmr}{r^2 + a^2}$$

is displayed in the accompanying figure.

Notice in particular the repulsive barrier of height $GMm/(2|a|)$ at $r = -|a|$. The lowest possible value for E_{class} is $-GMm/(2|a|)$, corresponding to a body at rest at $r = |a|$. For any energy in the range $-GMm/(2|a|) < E_{\text{class}} < 0$, the motion of the body will be confined between two finite positive values of r; namely,

$$r = |a| \left\{ \left(\frac{GMm/(2|a|)}{-E_{\text{class}}}\right) \pm \sqrt{\left(\frac{GMm/(2|a|)}{-E_{\text{class}}}\right)^2 - 1} \right\} .$$

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[2] We have already discussed the rationale behind this shortcut.

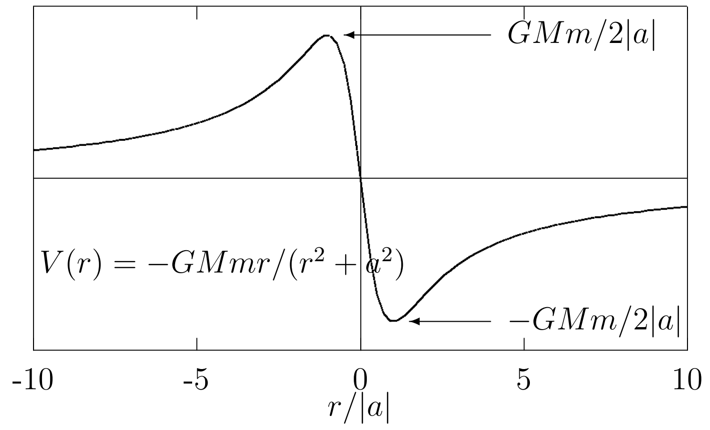

Figure 1: The effective potential for purely axial motion in Kerr's geometry.

For any energy in the range $0 < E_{\text{class}} < GMm/(2|a|)$, the motion of the body will be confined either to

$$r \geq -|a| \left\{ \left(\frac{GMm/(2|a|)}{E_{\text{class}}} \right) - \sqrt{\left(\frac{GMm/(2|a|)}{E_{\text{class}}} \right)^2 - 1} \right\}$$

or to

$$r \leq -|a| \left\{ \left(\frac{GMm/(2|a|)}{E_{\text{class}}} \right) + \sqrt{\left(\frac{GMm/(2|a|)}{E_{\text{class}}} \right)^2 - 1} \right\} .$$

Finally, for any energy in the range $E_{\text{class}} \geq GMm/(2|a|)$, the motion will be unconstrained. Whenever $E_{\text{class}} \geq 0$, the speed of the body at $r = \pm\infty$ is given by v_0, where

$$E_{\text{class}} = \frac{1}{2} m v_0^2,$$

i.e.,

$$E = mc^2 \sqrt{1 + (v_0/c)^2}.$$

It should be noted that in those cases where motion occurs on the negative r side of the potential barrier, the body moves ever more rapidly as $r \to -\infty$, but the acceleration continually decreases. In this region the Kerr object

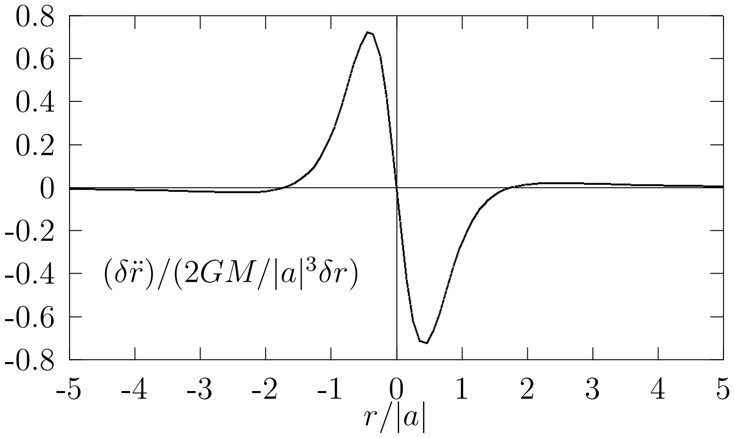

Figure 2: Graph of $(\delta\ddot{r})/(\delta r)$ versus r.

behaves as a negative gravitational mass. The same conclusion is reached if one notes that the full Kerr metric involves the first power of r only in the combination Mr.

The variation of $\ddot{r}$ with r is of no small interest, if one does not wish to be torn apart by tidal forces. By differentiating Eq. (12.2), we obtain the result

$$\delta\ddot{r} + \frac{2GMr(3a^2 - r^2)}{(r^2 + a^2)^3}\delta r = 0.$$

As is clear from the accompanying figure, the tension on a radially oriented falling body increases gradually as r decreases, reaching a peak at $r = (\sqrt{2} + 1)|a| \approx 2.414|a|$. At $r = \sqrt{3}|a| \approx 1.732|a|$ the tension is replaced by a compressive force that increases dramatically until $r = (\sqrt{2} - 1)|a| \approx 0.414|a|$. Falling rapidly the compressive force is again replaced by a tension, that reaches a maximum value at $r = -(\sqrt{2} - 1)|a| \approx -0.414|a|$, and so on.

If $|(\delta\ddot{r})/(\delta r)|$ is no larger than, say, 1 $\sec^{-2}$, then the variation in gravitational acceleration over objects of human size will probably[3] be less than 1g throughout the entire contemplated journey. However, the maximum value of $|(\delta\ddot{r})/(\delta r)|$ is approximately $1.4GM/|a|^3$. If, for example, $|a| = GM/c^2$, then $GM/|a|^3 = (c/|a|)^2$. Hence we wish to select a Kerr object for which $|a|/c$ is at least 1 sec., or $GM/c^2 \approx |a| \approx 1$ light-second, i.e., 300,000 km. On

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[3] The "probably" reflects the fact that we have not yet related δr to a real distance.

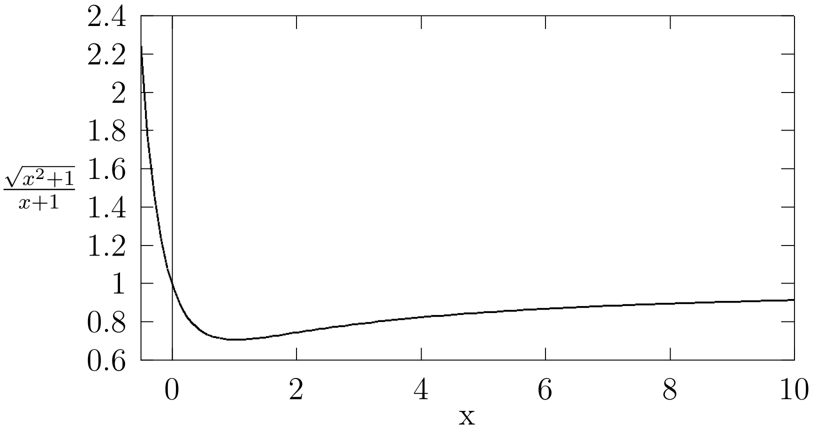

Figure 3: The integrand $\sqrt{x^2+1}/(x+1)$.

the other hand, we know $GM_\odot/c^2 \approx 1.4$ km. Thus, we should consider only journeys to rapidly rotating Kerr objects that are more massive than about $10^5 M_\odot$.[4]

Perhaps the most interesting journey to contemplate would be one in which the spaceship had an energy corresponding to

$$v_0^2 = GM/|a| \ ,$$

for in this case it would come to rest at the point of unstable equilibrium at $r = -|a|$. One could then decide whether to go ahead, or return. If the magnitude of a is much greater than GM/c^2, the required speed v_0 could be quite small compared to the speed of light, but if the magnitude of a is only slightly larger than GM/c^2, one will have to travel at near the speed of light to reach the top of the potential barrier. Moreover, you will accelerate until you reach $r = |a|$, then begin decelerating, coming to rest at $r = -|a|$. Of course, you won't feel the acceleration, as you will be in a state of free fall throughout the trip. Only tidal forces will be of real concern to you.

When $v_0^2 = GM/|a|$,

$$\dot{r} = \pm v_0 \frac{r + |a|}{\sqrt{r^2 + a^2}}.$$

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[4] The mass M could be reduced somewhat if the ratio $|a|/M$ were increased.

The proper time to travel from r_1 in to r_2 is then given by

$$v_0\tau = \int_{r_2}^{r_1} \frac{\sqrt{r^2+a^2}}{r+|a|}dr = |a|\int_{r_2/|a|}^{r_1/|a|} \frac{\sqrt{x^2+1}}{x+1}dx.$$

The behavior of the integrand is displayed in the accompanying figure. For $x >> 1$ the integrand is approximately equal to unity. It is also equal to unity at $x = 0$ and has a minimum value $1/\sqrt{2}$ at $x = 1$. From this we conclude that the proper time required to reach $r = 0$ from any given $r = r_{\text{init}} > 0$ will be approximately equal to r_{init}/v_0. For $x < 0$ the integrand rises dramatically, diverging as $x \to -1$. The proper time required to reach $r = -|a|$ is, in fact, infinite, the integral having a logarithmic divergence as $r \to -|a|$.

Actually, the integral can be performed analytically, with the result that, for $b > -1$,

$$\int_0^b \frac{\sqrt{x^2+1}}{x+1}dx = \sqrt{b^2+1} - \ln\left(\sqrt{b^2+1}+b\right) - \sqrt{2}\sinh^{-1}\left(\frac{1-b}{1+b}\right) - 1.$$

Perhaps it would not be amiss to reflect upon what is meant by your *speed.* The rate of change of r with respect to proper time τ is given by

$$\dot{r}^2 = v_0^2 + \frac{2GMr}{r^2+a^2}\,,$$

which becomes

$$\dot{r}^2 = \left(\frac{GM}{|a|}\right)\frac{(r+|a|)^2}{r^2+a^2}$$

when $v_0^2 = GM/|a|$ (where, of course, we assume that $|a| > GM/c^2$).

The function $(r+|a|)^2/(r^2+a^2)$ reaches its greatest value 2 at $r = |a|$. Hence, if $|a| < 2GM/c^2$, there will be a certain region in which $\dot{r}^2$ exceeds unity. It would appear that $\dot{r}$ cannot be your actual speed, since that can never exceed the speed of light.

We have been using the proper time τ, which is the time recorded by clocks on your ship (including your own heart beat), but we have also been using the radial coordinate r, which is just a label of a spacetime event. It may or may not have a direct interpretation in terms of distance along the axis, so there is no reason to believe $\dot{r}$ is your real speed, which can only be identified by carrying out a thought (or Gedanken[5]) experiment.

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[5] Another German word that is in common use.

Suppose we compare your progress with that of a radial light ray traveling in the same direction. For a light ray, proper time never evolves, i.e., $\delta\tau = 0$. Hence, events connected by a radial light ray have labels r, t such that

$$\left(\frac{dr}{dt}\right)^2_{\text{light}} = c^2\Delta^2 \ ,$$

where

$$\Delta := 1 - \frac{(2GM/c^2)r}{r^2 + a^2} \ .$$

On the other hand, events that are connected by your world line as you free fall along the axis have labels r, t such that

$$\left(\frac{dr}{dt}\right)^2 = c^2\Delta^2 \left(\frac{(E/mc^2)^2 - \Delta}{(E/mc^2)^2}\right) \ .$$

The ratio of these two expressions is

$$\frac{\left(\frac{dr}{dt}\right)^2}{\left(\frac{dr}{dt}\right)^2_{\text{light}}} = \frac{(E/mc^2)^2 - \Delta}{(E/mc^2)^2},$$

which is always less than unity, because $\Delta > 0$. From this analysis we conclude that your actual speed v is given by

$$v^2 = c^2 \frac{(E/mc^2)^2 - \Delta}{(E/mc^2)^2}.$$

This has been another good lesson in not attributing to the symbols that label spacetime events an unwarranted à priori physical or geometrical significance. Usually such significance can only be ascertained through Gedanken experiments, which may be exceedingly difficult to analyze, especially if the spacetime model is not asymptotically flat and if it has no symmetries.

It is also interesting that the last formula can be expressed in the form

$$v^2 = c^2 \frac{\dot{r}^2}{1 + v_0^2/c^2} \ .$$

For any given v_0, this suggests replacing the coordinate r by R, where

$$r = \sqrt{1 + v_0^2/c^2 R} \ .$$

A radially oriented meter stick in your freely falling rocket ship measures differences in R directly, while the r-values at the two ends of such a meter stick would differ by $\sqrt{1 + v_0^2/c^2}$ meters.

Nature of the coordinates r, θ, φ, t

Throughout the whole spacetime (with $|a| > GM/c^2$) the vector fields $\boldsymbol{\partial}_r$ and $\boldsymbol{\partial}_\theta$ are both spacelike. Moreover, throughout the region of spacetime that was explored by a body freely falling along the symmetry axis, the label t behaved as a good time variable should; namely, throughout the journey one had $\dot{t} > 0$ and $\boldsymbol{\partial}_t$ was timelike.

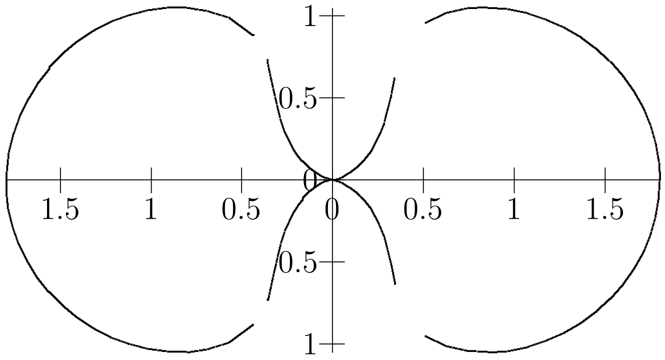

Figure 4: For $a = 1.10(GM/c^2)$ the region in which $g_{tt} > 0$ is shown in a polar plot, using r as the radial coordinate and θ as the polar angle. The $\theta = 0$ axis is directed upward. Remember that $r = 0$ is not a point and r can assume both positive and negative values.

On the other hand, as we already observed in the last chapter, there is a region (with $r > 0$) in which $g_{tt} > 0$. The boundary of this ergosphere is shown in the accompanying figure. Within the ergosphere the $\boldsymbol{\partial}_t$ is spacelike. It is instructive to compare the nature of the ergosphere for $|a| < GM/c^2$ that we displayed in the last chapter and the ergosphere for $|a| > GM/c^2$ that we displayed in this chapter. Even though both $\boldsymbol{\partial}_\varphi$ and $\boldsymbol{\partial}_t$ are spacelike within the ergosphere, there exist linear combinations of these two vector fields that are timelike. Therefore, one is not compelled to move toward ever decreasing values of r, and in fact it is possible to move off toward $r = \infty$.

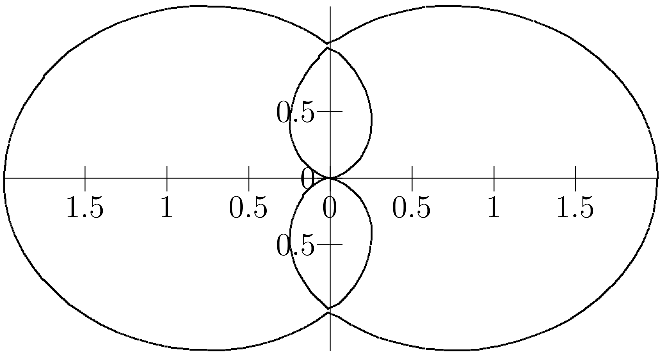

Figure 5: For the *extreme* case, $a = GM/c^2$, the region in which $g_{tt} > 0$ is shown.

Traveling backwards in time

We note that, in general,

$$\dot{t} = \omega\dot{\varphi} \pm f^{-1}\sqrt{f(1 + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2) + \rho^2\dot{\varphi}^2},$$

where

$$\begin{aligned} f &= 1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}, \\ \omega &= -\frac{2Mr\,a\sin^2\theta}{r^2 + a^2\cos^2\theta - 2Mr}, \\ \rho^2 &= \Delta\sin^2\theta. \end{aligned}$$

Let us suppose that we have managed, somehow, to move into an orbit (about a Kerr object with $GM/c^2 = 1$ and $a = 2$) for which

$$r = -0.4, \quad \theta = \frac{\pi}{2}, \quad \dot{r} = \dot{\theta} = 0, \quad \dot{\varphi} = -1.$$

For this orbit, $\Delta = 4.96$ and $\rho = \Delta^{1/2} = 2.227$ while $g_{\varphi t} = f\omega = 5$. Thus, $\rho < g_{\varphi t}$ and $g_{\varphi\varphi} = f^{-1}\rho^2 - f\omega^2 < 0$. We are in an *acausal* region where $\boldsymbol{\partial}_\varphi$ is timelike! The boundary of this region is determined by the quartic equation

$$(r^2 + a^2)(r^2 + a^2\cos^2\theta) + (2GM/c^2)a^2 r\sin^2\theta = 0,$$

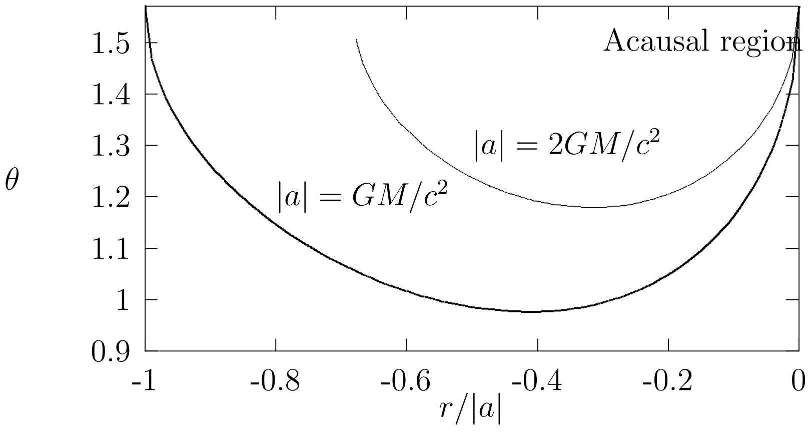

Figure 6: For the *extreme* case, $|a| = GM/c^2$, and for the case $|a| = 2GM/c^2$, the region in which $g_{\varphi\varphi} < 0$ is shown. Note that for larger values of $|a|$, the acausal region shrinks toward $r = 0$ and $\theta = \pi/2$.

which has real solutions when $r < 0$.

In the orbit I have selected, $f = 6$ and $\omega = 5/6$, while

$$\dot{t} = \frac{1}{6}\left\{5\dot{\varphi} \pm \sqrt{6 + 4.96\dot{\varphi}^2}\right\} = \left\{ \begin{array}{c} -0.2816 \\ -1.3851 \end{array} \right.$$

As usual, we take the larger root, $\dot{t} = -0.2816$, which is negative! The longer we stay in this orbit, the earlier it gets. If we can stay here long enough, we ought to be able to arrive home before we started our journey.

How fast are we moving in our present orbit? Anywhere within the Boyer-Lindquist chart we can employ the orthonormal tetrad of one-forms

$$\begin{aligned} e^1 &= \sqrt{\frac{r^2 + a^2\cos^2\theta}{\Delta}}dr, \\ e^2 &= \sqrt{r^2 + a^2\cos^2\theta}d\theta, \\ e^3 &= \frac{\sin\theta}{\sqrt{r^2 + a^2\cos^2\theta}}[(r^2 + a^2)d\varphi - adt], \\ e^4 &= \sqrt{\frac{\Delta}{r^2 + a^2\cos^2\theta}}(dt - a\sin^2\theta d\varphi). \end{aligned}$$

With respect to this frame the four-velocity has components

$$\begin{aligned}
V^1 &= \sqrt{\frac{r^2 + a^2\cos^2\theta}{\Delta}}\dot{r}, \\
V^2 &= \sqrt{r^2 + a^2\cos^2\theta}\,\dot{\theta}, \\
V^3 &= \frac{\sin\theta}{\sqrt{r^2 + a^2\cos^2\theta}}[(r^2 + a^2)\dot{\varphi} - a\dot{t}], \\
V^4 &= \sqrt{\frac{\Delta}{r^2 + a^2\cos^2\theta}}(\dot{t} - a\sin^2\theta\dot{\varphi}).
\end{aligned}$$

In the case being considered at this time, $V^1 = V^2 = 0$ and $V^3 = 8.99$ and $V^4 = 9.045$. To what actual speed does this correspond? Equating V^3 to $v/\sqrt{1-(v/c)^2}$, we find $v = 0.994$, just below the speed of light. As we saw in Ch. 6, it would take about three years to accelerate from rest up to this speed, and another three years to decelerate. While daunting, this alone would not discourage us from attempting to travel into the past.

Ex. 27 *Compute the orthonormal components of the four-velocity* $\mathbf{V}$ *for the freely falling traveler moving along the axis of the Kerr object. Show that the actual speed of the traveler that is determined this way agrees with that which we found by comparing his trajectory with that of a light ray.*

One thing that should concern us is the maximum g-force that we shall experience traveling in the acausal region. While a freely falling body is subjected to no g-forces at all, as we travel along our non-geodesic timelike orbit we shall experience[6] a g-force $-m\mathbf{A}$, where

$$\mathbf{A} = \left\{\frac{dV^k}{d\tau} + \Gamma_{ij}^{\ k} V^i V^j\right\}\mathbf{e}_k.$$

If one differentiates $\mathbf{V}\cdot\mathbf{V} = -c^2$ with respect to the proper time τ, one finds that

$$\mathbf{A}\cdot\mathbf{V} = g_{\mu\nu}A^\mu V^\nu = 0.$$

Therefore, in our rest frame, $\mathbf{A}$ only has spatial components. In order to proceed further we need to know the connection components $\Gamma_{ij}^{\ k}$ relative to our orthonormal tetrad.[7] A straightforward but tedious calculation reveals

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[6] The proper acceleration is $\mathbf{A}$ evaluated in that frame in which the spatial components of $\mathbf{V}$ and the temporal component of $\mathbf{A}$ all vanish.

[7] Remember that $\Gamma_{ji}^{\ k} \neq \Gamma_{ij}^{\ k}$ when one employs an orthonormal basis.

that

$$
\begin{aligned}
&(r^2+a^2\cos^2\theta)^{3/2}\Gamma_{ij}^{\;1} = \\
&\begin{pmatrix}
0 & -a^2\cos\theta\sin\theta & 0 & 0 \\
0 & -r\Delta^{1/2} & 0 & 0 \\
0 & 0 & -r\Delta^{1/2} & -ar\sin\theta \\
0 & 0 & -ar\sin\theta & \frac{M(r^2-a^2\cos^2\theta)-ra^2\sin^2\theta}{\Delta^{1/2}}
\end{pmatrix}, \\
&(r^2+a^2\cos^2\theta)^{3/2}\Gamma_{ij}^{\;2} = \\
&\begin{pmatrix}
a^2\cos\theta\sin\theta & 0 & 0 & 0 \\
r\Delta^{1/2} & 0 & 0 & 0 \\
0 & 0 & -(r^2+a^2)\cot\theta & -a\Delta^{1/2}\cos\theta \\
0 & 0 & -a\Delta^{1/2}\cos\theta & -a^2\cos\theta\sin\theta
\end{pmatrix}, \\
&(r^2+a^2\cos^2\theta)^{3/2}\Gamma_{ij}^{\;3} = \\
&\begin{pmatrix}
0 & 0 & 0 & -ar\sin\theta \\
0 & 0 & 0 & a\Delta^{1/2}\cos\theta \\
r\Delta^{1/2} & (r^2+a^2)\cot\theta & 0 & 0 \\
ar\sin\theta & a\Delta^{1/2}\cos\theta & 0 & 0
\end{pmatrix}, \\
&(r^2+a^2\cos^2\theta)^{3/2}\Gamma_{ij}^{\;0} = \\
&\begin{pmatrix}
0 & 0 & -ar\sin\theta & 0 \\
0 & 0 & a\Delta^{1/2}\cos\theta & 0 \\
-ar\sin\theta & -a\Delta^{1/2}\cos\theta & 0 & 0 \\
\frac{M(r^2-a^2\cos^2\theta)-ra^2\sin^2\theta}{\Delta^{1/2}} & -a^2\cos\theta\sin\theta & 0 & 0
\end{pmatrix}.
\end{aligned}
$$

In our case we find that we are subjected to a centrifugal[8] force, for $A^2 = A^3 = A^0 = 0$ and

$$A^1 = -\frac{(\Delta^{1/2}V^3 + aV^0)^2 + Mr(V^0)^2}{r^2\Delta^{1/2}} = -3,984.$$

Now, what does this number mean in terms of the number of g's that we shall experience. Remember that 1g is equal to 9.8 m/sec^2. Therefore, approximately,

$$1\text{g} \;=\; 10\text{m/sec}^2 = 10^{-2}\text{km/sec}^2 = 10^{-2}\left(\frac{c}{3\times 10^5\text{km/sec}}\right)^2 \text{km/sec}^2$$

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[8] The space ship will have to exert a force upon our bodies that is directed radially inward, a centripetal force, but we shall describe this as our being subjected to an outward (or centrifugal) g-force

$$\begin{aligned} &= \frac{1}{9}\times 10^{-12}c^2\text{km}^{-1} = \frac{1}{9}\left(\frac{1.5\text{km}}{GM_\odot/c^2}\right)\times 10^{-12}c^2\text{km}^{-1} \\ &= 0.17\times 10^{-12}c^2\left(GM_\odot/c^2\right)^{-1} = 0.17\times 10^{-12}\left(M/M_\odot\right), \end{aligned}$$

since $c = 1$ and $GM/c^2 = 1$ in the units we have been employing. We conclude that the number of g's that we shall experience is about $2\times 10^{16}\frac{M_\odot}{M}$. Thus, such an orbit is only feasible for humans if the Kerr object has a mass that is more than about 10^{16} solar masses. Our own Milky Way galaxy has a mass that is about 10^{11} solar masses, so we would require a Kerr object with a mass equal to that of 10^5 galaxies.

It is unlikely that using some other orbit in the acausal region would dramatically lower the amount of mass that would be necessary in order to make travel into the past feasible for humans. Nevertheless, I would encourage you to try to find a significantly more practical orbit than the one I tried as a test case.

Other exact solutions of the Einstein equations have been discovered that possess closed timelike orbits and hence enable time travel into the past. Naturally, any travel into one's own past presents one with paradoxes that are not easily resolved without suspending the laws of physics. For this reason people have entertained various conjectures, the latest of which is Stephen Hawking's "Chronology Protection Conjecture," which, if true, would not permit a solution of the Einstein field equations with "reasonable" source terms and nonsingular initial data to evolve into a solution with closed timelike orbits. At this time it is not yet known precisely how to formulate the conjecture. That is the problem with conjectures, before they become theorems that are subject to being proved mathematically. Of course, if an exact solution were to be found that involved "reasonable" source terms and which nevertheless evolved from nonsingular initial data to a state involving closed timelike orbits, one would begin to suspect that the idea of "Chronology Protection" is simply wrong. At this time, however, the situation remains unclear.

It would be the supreme irony if the one universe of which we are a minute part really did possess closed timelike orbits that will ultimately make ridiculous the notions of cause and effect upon which centuries of fruitful science have been based.