Chapter 11

Kerr's Rotating Schwarzschild Solution

One of the most exciting discoveries in the history of relativity took place in 1963, when R. P. Kerr published his fascinating new solution of the vacuum Einstein field equations.[1] The discovery of the Kerr metric was the serendipitous result of a study of vacuum spacetimes that have algebraically special curvature tensors, not the result of an attempt to solve directly the general equations that govern any stationary axisymmetric spacetime.

The paper in which Kerr announced his discovery was only one and one-half pages long, and many years elapsed before there was a follow up paper in which more details of the derivation were provided. At the time very few relativists were comfortable using the tetrad formulation of general relativity, and verifying that Kerr had a solution of the vacuum field equations by more traditional means was difficult indeed.

Kerr expressed his solution in the form[2]

$$\begin{aligned} ds^2 &= (r^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta\, d\phi^2) \\ &\quad +2(du - a\sin^2\theta\, d\phi)(dr - a\sin^2\theta\, d\phi) \\ &\quad -\left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right)(du - a\sin^2\theta\, d\phi)^2\ . \end{aligned} \tag{11.1}$$

I should prefer, for the present, to employ another chart due to Boyer and Linquist, in terms of which the metric assumes the form

$$ds^2 = (r^2 + a^2\cos^2\theta)(d\theta^2 + \frac{dr^2}{r^2 + a^2 - 2Mr})$$

[1] R. P. Kerr, who is at least as well known as a world-class bridge player, published his remarkable discovery in Phys. Rev. Lett. **11**, 237 (1963).

[2] The sign of a has been changed to bring the notation into accord with current usage.

$$+(r^2+a^2\cos^2\theta)^{-1}\{\sin^2\theta[(r^2+a^2)d\varphi-adt]^2$$
$$-(r^2+a^2-2Mr)(dt-a\sin^2\theta d\varphi)^2\}. \qquad (11.2)$$

You can see that this solution reduces to the Schwarzschild solution when $a \to 0$. More generally, the metric is singular when $r^2+a^2\cos^2\theta=0$. When $a \neq 0$, this singularity is at $r=0, \theta=\pi/2$ in the equatorial plane. A computation of the curvature tensor confirms that this is a true singularity, rather than just the boundary of the chart.

We shall accept the validity of the Kerr solution without proof, as I should like you to see some of the attributes of this spacetime without delay. Don't be surprised if it takes you some time to show that Einstein's vacuum field equations are indeed satisfied! In another book we shall show how the exercise was much simplified through the introduction of the author's "complex potential" reformulation[3] of the Einstein equations.

The Equations of Motion

The Lagrangian from which the equations of motion of a test particle can be derived has the form

$$\begin{aligned} L &= (r^2+a^2\cos^2\theta)^{-1} \\ &\quad \left\{(r^2+a^2-2Mr)(\dot t-a\sin^2\theta\dot\varphi)^2-\sin^2\theta[(r^2+a^2)\dot\varphi-a\dot t]^2\right\} \\ &\quad -(r^2+a^2\cos^2\theta)\left(\dot\theta^2+\frac{\dot r^2}{r^2+a^2-2Mr}\right). \end{aligned}$$

Because L is independent of φ and t, there are two conserved quantities, the angular momentum ℓ about the symmetry axis, and the total energy E. If m is the mass of the test particle, these are given by

$$\begin{aligned} \ell &= m\left\{(r^2+a^2)\sin^2\theta\dot\varphi-\frac{2Mr\,a\sin^2\theta}{r^2+a^2\cos^2\theta}(\dot t-a\sin^2\theta\dot\varphi)\right\}, \\ E &= m\left\{\dot t-\frac{2Mr}{r^2+a^2\cos^2\theta}(\dot t-a\sin^2\theta\dot\varphi)\right\}, \end{aligned}$$

respectively. Taking appropriate linear combinations of the two conserved quantities, we find that

$$\begin{aligned} \ell-a\sin^2\theta E &= m\sin^2\theta[(r^2+a^2)\dot\varphi-a\dot t], \\ E-\frac{a}{r^2+a^2}\,\ell &= m\left(1-\frac{2Mr}{r^2+a^2}\right)(\dot t-a\sin^2\theta\dot\varphi). \end{aligned}$$

[3] F. J. Ernst, Phys. Rev. **167**, 1175 (1968); Phys. Rev. **168**, 1415 (1968).

It should be noted that

$$\begin{aligned} \dot t - a\sin^2\theta\dot\varphi & \quad \text{is a function of } r \text{ alone,} \\ (r^2+a^2)\dot\varphi - a\dot t & \quad \text{is a function of } \theta \text{ alone,} \end{aligned}$$

and when $a \to 0$, ℓ and E reduce to the values we obtained for the Schwarzschild metric.

Because the proper time τ does not appear explicitly in the Lagrangian, the Hamiltonian is also a constant of the motion, and because the Lagrangian is homogeneous quadratic in the generalized velocities, the Hamiltonian is equal to the Lagrangian, which is therefore also a conserved quantity. By rescaling the proper time, we can rescale the Lagrangian to unity, as we did earlier in the Schwarzschild case. Because of the existence of what is known as a *Killing tensor*, the Kerr metric admits four constants of the motion rather than just three.[4] We shall denote the fourth constant of the motion by K, and write the remaining equations of motion in the form

$$\begin{aligned} \left[(\frac{r^2+a^2\cos^2\theta}{r^2+a^2-2Mr})\dot r\right]^2 - \left[\frac{E-\frac{a}{r^2+a^2}\ell}{m(1-\frac{2Mr}{r^2+a^2})}\right]^2 & \\ +\frac{r^2+K^2}{r^2+a^2-2Mr} &= 0\ , \\ \left[(r^2+a^2\cos^2\theta)\dot\theta\right]^2 + \left[\frac{(\ell/\sin\theta)-Ea\sin\theta}{m}\right]^2 & \\ +a^2\cos^2\theta - K^2 &= 0\ . \end{aligned}$$

Motion in the Equatorial Plane

The equatorial plane corresponds to $\theta = \pi/2$. If initially $\theta = \pi/2$ and $\dot\theta = 0$, then the constant K is given by

$$K = \frac{\ell - aE}{m}\ .$$

With this value of K, however, $\ddot\theta = 0$ too, so the moving body will remain in the equatorial plane just as in the classical Kepler problem. Moreover, if an object is released from rest at a large positive value of r, then $\ell = 0$ and

[4]Such Killing tensors were discussed by I. Hauser and R. J. Malhiot in J. Math. Phys. **16**, 150, 1625 (1975).

$E \approx m$. For such motion in the equatorial plane, one finds

$$\dot{\varphi} = \frac{a}{r^2 + a^2 - 2Mr} \frac{2M}{r}$$

and

$$\dot{t} = 1 + \frac{2M}{r}\left(\frac{r^2 + a^2}{r^2 + a^2 - 2Mr}\right).$$

Thus, a falling body is dragged by the gravitational field of the rotating Kerr object. As one approaches the boundary of the Boyer-Lindquist chart,

$$r_+ := M + \sqrt{M^2 - a^2}\ ,$$

where $r^2 + a^2 - 2Mr = 0$, this dragging effect increases without limit.[5] Similarly, $\dot{t}$, which measures the time-dilation effect, increases without limit.

But wait! Can we really consider the coordinate t to have anything to do with time when

$$M - \sqrt{M^2 - a^2\cos^2\theta} \le r \le M + \sqrt{M^2 - a^2\cos^2\theta},$$

the region in which $g_{tt} \ge 0$? Inside this region, bounded by the outer and inner infinite redshift surfaces, shown in the accompanying figure, the field $\boldsymbol{\partial}_t$ is spacelike!

In the Schwarzschild case, the outer horizon and the infinite red-shift surface were one and the same surface. In the Kerr case, these are separated by a region called the *ergosphere*, where both $\boldsymbol{\partial}_\varphi$ and $\boldsymbol{\partial}_t$ are spacelike.

Ex. 25 *Show that when $\rho > 0$ it is possible to construct a linear combination of $\boldsymbol{\partial}_\varphi$ and $\boldsymbol{\partial}_t$ that is timelike.*

For this reason, it is possible, in principle, to move in such a way that r and θ remain constant. In fact, within the ergosphere one can move in such a way that r increases. One is in no way trapped as a result of entering the ergosphere of the Kerr metric.

Again restricting attention to motion in the equatorial plane, we note that in the Kepler problem conservation of energy and conservation of angular momentum are described by the respective equations

$$\begin{aligned} \ell &= mr^2\dot{\theta}^2\ , \\ E_{\text{class}} &= \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2] - \frac{GMm}{r}\ . \end{aligned}$$

[5] In this discussion we have tacitly assumed that $a^2 < M^2$. The Kerr object with $a^2 > M^2$ is not a black hole.

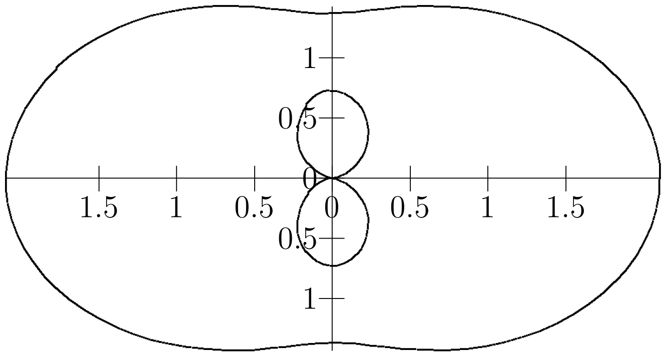

Figure 1: For $a = 0.95(GM/c^2)$ the region in which $g_{tt} > 0$ is shown in a polar plot using r as the radial coordinate and θ as the polar angle. The $\theta = 0$ axis is directed upward. Remember that $r = 0$ is not actually a point, and r can assume negative values.

Thus, in the Kepler problem,

$$\frac{E_{\text{class}}}{m} = \frac{1}{2}\left[\dot{r}^2 + \left(\frac{\ell}{mr}\right)^2\right] - \frac{GM}{r} ,$$

or

$$r^2\dot{r}^2 = (2E_{\text{class}}/m)r^2 + 2GMr - K^2 ,$$

where $K := \ell/m$. The minimum and maximum values of r, if any, satisfy the quadratic equation

$$(2E_{\text{class}}/m)r^2 + 2GMr - K^2 = 0 .$$

The two roots are given by

$$r = \frac{(GM/K) \pm \sqrt{(GM/K)^2 - 2(-E_{\text{class}}/m)}}{2(-E_{\text{class}}/m)} K .$$

Clearly, for the orbit to be circular, one must have

$$\begin{aligned} \frac{E_{\text{class}}}{m} &= -\frac{GM}{2r} , \\ K^2 &= GMr , \end{aligned}$$

or

$$\left(\frac{E_{\text{class}}}{m}\right) K^2 = -\frac{1}{2}(GM)^2 \ .$$

We shall now develop the analogous equations for the Kerr problem, where we use units such that $G = 1$, and the equations of motion assume the form

$$\begin{aligned} \ell - aE &= m[(r^2 + a^2)\dot{\varphi} - a\dot{t}] \ , \\ E - \frac{a}{r^2 + a^2}\,\ell &= m\left(1 - \frac{2Mr}{r^2 + a^2}\right)(\dot{t} - a\dot{\varphi}) \ , \end{aligned}$$

and

$$(r^2 + a^2)^2 \dot{r}^2 = \big((E/m)r^2 - aK\big)^2 - (r^2 + K^2)(r^2 + a^2 - 2Mr) \ ,$$

where

$$K = \frac{\ell - Ea}{m} \ .$$

The minimum and maximum values of r, if any, satisfy the quartic equation

$$\big[(E/m)r^2 - aK\big]^2 - (r^2 + K^2)(r^2 + a^2 - 2Mr) = 0 \ ,$$

which has one obvious root at $r = 0$. Avoiding the singularity at $r = 0, \theta = \pi/2$, we shall consider the other roots, which satisfy the cubic equation

$$\alpha r^3 + 2Kr^2 - \beta K^2 r + 2K^3 = 0 \ ,$$

where

$$\begin{aligned} \alpha &:= (K/M)[(E/m)^2 - 1] \ , \\ \beta &:= [2(E/m)aK + a^2 + K^2]/(KM) \ . \end{aligned}$$

The r-equation has at least one real root, and may have three. For a circular orbit two roots must be coincident. In this case, not only is the above cubic equation satisfied, but also the quadratic equation that is obtained by differentiating with respect to r is satisfied. This quadratic equation has the form

$$3\alpha r^2 + 4Kr - \beta K^2 = 0 \ .$$

Using the quadratic equation to eliminate the r^3 term from the cubic equation, we get another quadratic equation,

$$r^2 - \beta K r + 3K^2 = 0 \ .$$

The second of these equations can be used to calculate β when r/K is stipulated, and the first of these equations can be used to calculate α. The result is

$$\begin{aligned} \alpha &= x^{-3} - x^{-1} , \\ \beta &= x + 3x^{-1} , \\ x &:= r/K . \end{aligned}$$

Alternatively, the r^2 term can be eliminated from the two quadratic equations, which results in a linear equation that gives

$$r = \frac{\beta + 9\alpha}{3\alpha\beta + 4} K .$$

This value of r can now be substituted back into either quadratic equation to obtain the following relation that α and β must satisfy in order to have such a circular orbit:

$$\alpha\beta^3 + (\beta^2 - 19\alpha\beta - 27\alpha^2) - 16 = 0 .$$

Ex. 26 *Verify that when $a = 0$ and $x := r/|K| \gg 1$, the relativistic problem reduces to the Kepler problem considered earlier. Note that when $E \approx m$, one can introduce the "classical" total energy $E_{class} := \frac{1}{2}[(E/m)^2 - 1]m$.*

The Rotating Black Hole

Like Eddington's coordinates for the Schwarzschild solution, Kerr's original $\{u, r, \theta, \phi\}$ coordinates permit one to extend the spacetime across the event horizon $r = r_+$ to values of r that are less than r_+. In fact, both u and r run from $-\infty$ to $+\infty$. A discussion of the interior of the Kerr black hole can be found in a paper by B. Carter[6] It is also described extensively in the book by S. Hawking and G. Ellis.[7]

While it is possible to proceed all the way in to the singularity at $r = 0$ when $\theta = \pi/2$, one should not jump to the conclusion that the singularity of the Kerr metric is at a point. Indeed, the proper circumference of a circle $r =$ constant, $\theta = \pi/2$, is given by

$$2\pi R = 2\pi\sqrt{g_{\varphi\varphi}} = 2\pi\sqrt{r^2 + a^2 + \frac{2Ma^2}{r}} ,$$

[6] B. Carter, Phys. Rev. **174**, 1559 (1968).

[7] S. Hawking and G. Ellis, *The Large Scale Structure of Spacetime*, Cambridge University Press (1973).

so there is a minimum circumference circle with

$$r = (Ma^2)^{1/3} \ ,$$

for which the proper circumference is

$$2\pi R = 2\pi|a|\sqrt{1 + 3\left(\frac{M}{|a|}\right)^{2/3}} \ .$$

The proper circumference becomes infinite as $r \to 0$ with $\theta = \pi/2$. The Kerr singularity is not a point singularity but rather a ring singularity. If $\theta \neq \pi/2$, nothing prevents one from passing through the ring to regions in which $r < 0$.