Chapter 10

Voyage to a Black Hole

Soon after the discovery of the Schwarzschild solution, Einstein showed that it would be impossible to have a static spherically symmetric source so small that it would fit within the surface $r = 2GM/c^2$. Lots of strange things happen as one approaches the surface $r = 2GM/c^2$. For example, as one can plainly see from Eq. (9.3), there are circular light ray orbits of radius $3GM/c^2$ about the central mass. Moreover, the light emitted by atoms at $r = 2GM/c^2$ suffers an infinite wavelength shift toward the red. One might well wonder if it would be at all possible to send light rays (or radio signals) from very small r out toward infinity.

Radial orbits

We shall begin our study of Schwarzschild spacetime by considering *radial* paths for particles and light rays. Let us first consider the *Newtonian* treatment of a test body falling from rest from a point a distance R from the center of an inverse square gravitational field. By conservation of energy, we have

$$\frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} = E = -\frac{GMm}{R} \ ,$$

from which we can calculate the time required for the body to reach $r = 0$; namely,

$$\Delta t = \sqrt{\frac{R}{r_0 c^2}} \int_0^R \frac{r^{1/2} dr}{\sqrt{R - r}}.$$

This integral can be evaluated easily by introducing a new variable of integration, θ, such that $r = R\cos^2\theta$. Then, we obtain

$$\begin{aligned} \Delta t &= 2\sqrt{\frac{R^3}{r_0 c^2}} \int_0^{\frac{\pi}{2}} d\theta\ \cos^2\theta \\ &= \frac{\pi}{2}\sqrt{\frac{R^3}{r_0 c^2}} = \frac{\pi}{2\sqrt{2}}\sqrt{\frac{R^3}{GM}} \\ &= \frac{P}{4\sqrt{2}} \approx 0.18P, \end{aligned}$$

where P is the orbital period, as given by Kepler's third law, in a circular orbit of radius R about a central mass M. The time Δt to fall from $R \gg 2GM/c^2$ to $r = r_0 := 2GM/c^2$ is slightly less than this.

The relativistic calculation is carried out in a similar way, beginning with the relativistic expression of conservation of energy; namely,

$$\left(\frac{\dot r}{c}\right)^2 - \frac{r_0}{r} = \left(\frac{E}{mc^2}\right)^2 - 1 = -\frac{r_0}{R}\ ,$$

where the dot now denotes differentiation with respect to the proper time τ, and $r_0 = 2GM/c^2$. Eliminating r_0, we end up with the same equation

$$\frac{1}{2}m\dot r^2 - \frac{GMm}{r} = -\frac{GMm}{R}$$

that we solved in the Newtonian treatment. Hence the proper time required to reach $r = r_0 = 2GM/c^2$ is finite! Thus, if a person were to fall from rest toward the source of the Schwarzschild field, he would reach $r = 2GM/c^2$ after a finite length of time $\Delta\tau$ had elapsed on his watch. Two questions arise:

(a) How will distant observers describe the traveler's journey?

(b) What will this traveler experience as he approaches $r = 2GM/c^2$?

Using the equation

$$E = mc^2(1 - \frac{2GM}{c^2 r})\dot t\ ,$$

one can calculate the elapsed coordinate time Δt; namely,

$$\Delta t = \sqrt{\frac{R - r_0}{r_0 c^2}} \int_{r_0}^{R} \frac{r^{3/2}dr}{(r - r_0)\sqrt{R - r}}.$$

Unlike the integral for the elapsed proper time $\Delta\tau$, the integral for the elapsed coordinate time Δt diverges logarithmically at the lower limit. Thus, $\Delta t = \infty$. Distant observers would assert that the traveler never actually reached $r = 2GM/c^2$, but only approached that place asymptotically as $t \to \infty$. On the other hand, the traveler himself reached there after only a short time had elapsed on his own watch (or, if you prefer, after only a finite number of his own heart beats).

Beyond the event horizon

Looking at the various orthonormal components of the Riemann curvature tensor, which we had to evaluate in the course of working out the vacuum Einstein field equations, we see that none of the components diverges as one approaches $r = 2GM/c^2$. If one takes the expressions seriously for smaller values of r, one finds that only at $r = 0$ do certain orthonormal components of the Riemann curvature tensor become infinite. Thus, it seems likely that the difficulties in the metric at $r = 2GM/c^2$ are a manifestation of the r, θ, φ, t coordinate system breaking down; i.e., $r = 2GM/c^2$ is the boundary of the r, θ, φ, t chart!

We shall continue to suppress the θ, φ coordinates; i.e., restrict attention to radial paths. Then the proper time is given by

$$\begin{aligned} d\tau^2 &= (1 - \frac{2GM}{c^2 r})dt^2 - \frac{1}{c^2}\frac{dr^2}{1 - \frac{2GM}{c^2 r}} \\ &= (1 - \frac{2GM}{c^2 r})\{dt - \frac{1}{c}\frac{dr}{1 - \frac{2GM}{c^2 r}}\}\{dt + \frac{1}{c}\frac{dr}{1 - \frac{2GM}{c^2 r}}\} . \end{aligned}$$

Eddington was the first to note that one could pass beyond the edge of the Schwarzschild chart by introducing a new chart r, θ, φ, u, where the "retarded time" u is defined by

$$u = t - \frac{1}{c}\int \frac{dr}{1 - \frac{2GM}{c^2 r}} .$$

In terms of the new chart,

$$d\tau^2 = (1 - \frac{2GM}{c^2 r})du^2 + \frac{2}{c}dr\, du .$$

Here there is no problem as $r \to 2GM/c^2$. The range of the coordinate u is $-\infty$ to $+\infty$. The curves $\theta = \text{const}, \varphi = \text{const}, u = \text{const}$ correspond to radially outgoing light rays. Holding θ, φ fixed, the equation $u = \text{const}$

describes an outgoing radial light ray that may be emitted at $r = 0$, proceed to $r > 2GM/c^2$ and eventually to $r = \infty$.

One can introduce a third coordinate chart r, θ, φ, v, where v is the "advanced time"

$$v = t + \frac{1}{c} \int \frac{dr}{1 - \frac{2GM}{c^2 r}} \; .$$

In terms of this chart,

$$d\tau^2 = (1 - \frac{2GM}{c^2 r}) dv^2 - \frac{2}{c} dr \; dv \; .$$

Holding θ, φ fixed, the equation $v = \text{const}$ describes an incoming radial light ray that will ultimately pass to $r < 2GM/c^2$ and then to $r = 0$. If one is interested in describing a journey into a black hole, then r, θ, φ, v provides one suitable coordinate system. This permits one to cross the *future event horizon.* Neither the original Schwarzschild coordinate system nor either of the Eddington coordinate systems cover the whole of the extended Schwarzschild spacetime.

A chart that provides a maximal extension of Schwarzschild spacetime was discovered by Kruskal. In effect, he introduced both an advanced and a retarded time coordinate, abandoning r as well as t. Let us provisionally introduce a chart u', v', θ, φ such that

$$\begin{aligned} du' &= dt - \frac{1}{c} \frac{dr}{1 - \frac{r_0}{r}} \; , \\ dv' &= dt + \frac{1}{c} \frac{dr}{1 - \frac{r_0}{r}} \; . \end{aligned}$$

The coordinates u' and v' run from $-\infty$ to $+\infty$. Clearly

$$t = \frac{1}{2}(u' + v') \; ,$$

while r can be found by solving the transcendental equation

$$r + r_0 \ln (\frac{r - r_0}{r_0}) = \frac{c}{2}(v' - u') \; .$$

Consider the following alternative way of expressing these equations:

$$\exp (\frac{r}{r_0}) \times (\frac{r - r_0}{r_0}) = \exp (-\frac{cu'}{2r_0}) \times \exp (\frac{cv'}{2r_0}) > 0$$

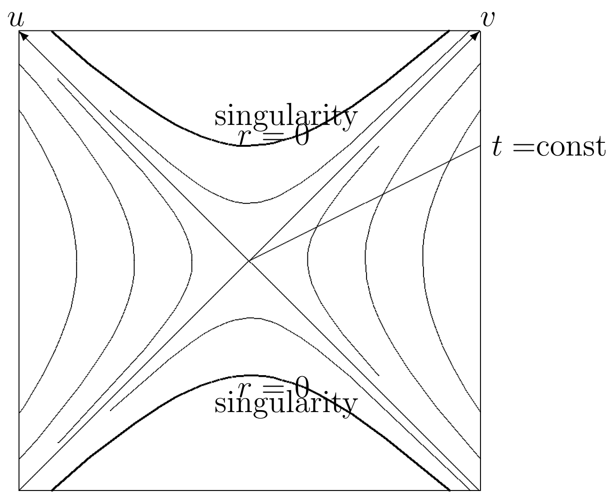


Figure 1: Kruskal diagram: Lines r = const are hyperbolas, while lines t = const are straight lines through the origin. The future horizon ($r = r_0$, $t = +\infty$) is the straight line at 45 degrees, while the past horizon ($r = r_0$, $t = -\infty$) is the straight line at -45 degrees. The only curvature singularities are at $r = 0$.

and

$$\exp\left(\frac{ct}{r_0}\right) = \exp\left(\frac{cu'}{2r_0}\right) \times \exp\left(\frac{cv'}{2r_0}\right) > 0 \ .$$

Let us now replace the provisional coordinates u', v' by our final choice u, v, where

$$\begin{aligned} u &:= -\exp\left(-cu'/2r_0\right) \ , \\ v &:= \exp\left(cv'/2r_0\right) \ , \end{aligned}$$

and where $u < 0$ and $v > 0$. We then have

$$\left(\frac{r - r_0}{r_0}\right) \exp\left(\frac{r}{r_0}\right) = -uv$$

and

$$\exp\left(\frac{ct}{r_0}\right) = -\frac{v}{u} \ .$$

In these last equations, we may extend the chart to include non-negative values of u and non-positive values of v, so it is possible to reach $r < r_0$.

The metric can now be expressed in the form

$$d\tau^2 = 2f(u,v)^2 \, du \, dv - \frac{r(u,v)^2}{c^2}[d\theta^2 + \sin^2\theta d\varphi^2] \ ,$$

where a transcendental equation for $r(u, v)$ has already been given, and where $f(u, v)$ is determined from

$$f^2 = \frac{4r_0^3}{c^2 r} \exp\left(-\frac{r}{r_0}\right) \ .$$

The u, v, θ, φ coordinate chart covers the whole spacetime. The portion in which $u < 0$ and $v > 0$ corresponds to the portion covered by the original Schwarzschild chart r, θ, φ, t. It has become customary to plot a spacetime diagram with lines of constant u inclined at +45 degrees, and lines of constant v inclined at -45 degrees. The former correspond to radially outgoing light rays, the latter to radially incoming light rays. Plotted on such a spacetime diagram, lines of constant t are straight lines $v/u = \text{const}$ passing through the origin, while lines of constant r are hyperboloids $uv = \text{const}$ with the straight lines $u = 0$ and $v = 0$ as asymptotes. The only curvature singularities occur on the two sheets of the hyperboloid $uv = 1$, corresponding to $r = 0$.

Using the Kruskal diagram, it is easy to see what will transpire if an intrepid voyager falls into the region $r < r_0$. The straight line upon which

$t = \infty$ coincides with part of the degenerate hyperboloid corresponding to $r = r_0$. This is the reason why Schwarzschild's original chart broke down at $r = r_0$. Suppose at $t = 0$ an observer at $r = 10r_0$ begins to fall radially inward toward smaller r values. His path must lie entirely within the "forward light cone," since he cannot travel as fast as the speed of light. After a finite proper time has elapsed he approaches $r = r_0$, but t is then approaching ∞. Suppose the falling voyager sends a radio message back home. It will take longer and longer (in t time) for the message to reach $r = 10r_0$. Any message sent after he reaches $r = r_0$ will never reach $r = 10r_0$, but instead it will eventually reach the curvature singularity at $r = 0$. All radio signals go "inward." Moreover, since the hapless voyager must travel a world line that is within the forward light cone, he too will inevitably reach $r = 0$, where he will be infinitely compressed in one direction, infinitely expanded in another, by the infinite tidal forces associated with the singularity in spacetime. It is believed that stars that are much more massive than the Sun may ultimately have a fate in which the surface of the star may collapse within $r = r_0$, so that a "black hole" can be realized in nature.

Relativistic and classical orbits

Now we should like to develop further the comparison of relativistic and classical orbits that we initiated in the last chapter. Except for the u^3 term the relativistic orbital equation

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 - \beta^2 u - u^3 = \alpha$$

of the Einstein theory is identical to the classical equation. Here, the dependent variable is

$$u := \frac{2GM}{c^2 r} \ ,$$

and the energy and angular momentum parameters α and β are given by

$$\begin{aligned} \alpha &:= \beta^2 (\frac{2E_{\text{class}}}{mc^2}) \ , \\ \beta &:= \frac{2GMm}{c\ell} \ , \end{aligned}$$

where we have introduced an effective *classical energy*

$$E_{\text{class}} := \frac{1}{2} mc^2 [(E/mc^2)^2 - 1] \ .$$

As long as the entire orbit lies well outside the event horizon at $u = 1$, the u^3 term can be neglected completely, or taken into account using first order perturbation theory. In the absence of the u^3 term the orbital equation reduces to the Newtonian one, with its familiar conic section solutions. For example, if $\alpha = 0$, the solution is a parabola described by the equation

$$u = \beta^2 \sin^2(\frac{\varphi}{2}) \ ,$$

where the integration constant has been chosen so that $u = 0$ when $\varphi = 0$, i.e., $\varphi = 0$ when $r = \infty$. Clearly, u returns to 0 when $\varphi = 2\pi$, and $\varphi = \pi$ at closest approach, i.e., at $r = (2GM/c^2)\beta^{-2}$. Similarly, when $\alpha < 0$ the solutions are ellipses, and when $\alpha > 0$ the solutions are hyperbolas.

The exact solution of the relativistic equation can be expressed in terms of elliptic functions.[1] However, the main features of the orbits can be gleaned in the following manner. The cubic polynomial

$$f(u) := u^2 - \beta^2 u - u^3$$

that one has in the relativistic case can be compared with the quadratic polynomial

$$f_{\text{class}}(u) := u^2 - \beta^2 u$$

that one has in the classic Kepler problem. In the accompanying figures are shown these polynomials for $\beta = 0.43$, $\beta = 0.5$ and $\beta = 0.57$.

For motion to be possible the curve $f(u)$ (or $f_{\text{class}}(u)$) must lie below the horizontal line $\alpha = \text{const}$. In the standard Kepler problem the orbits are ellipses when $\alpha < 0$, providing α is at least the value necessary for a stable circular orbit. In this case the perihelion and aphelion distances can both be read off the graph.[2] When $\alpha > 0$ the orbits are hyperbolas. The boundary case $\alpha = 0$ corresponds to parabolic orbits. In each case the perihelion distance corresponding to a given value of α can be read off the graph.

In the relativistic case, for $\beta^2 < 1/3$ there is a range of values of $\alpha < 0$ for which bound orbits (analogous to the elliptical orbits of the Newtonian problem) exist. For each given value of $\beta^2 < 1/3$, there is a highest value of α for which a perihelion exists. For greater values of α a body will necessarily plunge into the black hole, the horizon of which is located at $u = 1$. For $\beta^2 = 1/3$ there is a minimum radius (unstable) circular orbit at $u = 1/3$,

[1] E. Jahnke and F. Emde, *Tables of Functions*, Dover (1945).

[2] Remember that $u := r_0/r$, so the perihelion corresponds to the highest value of u that is attained while the aphelion corresponds to the least value of u that is attained.

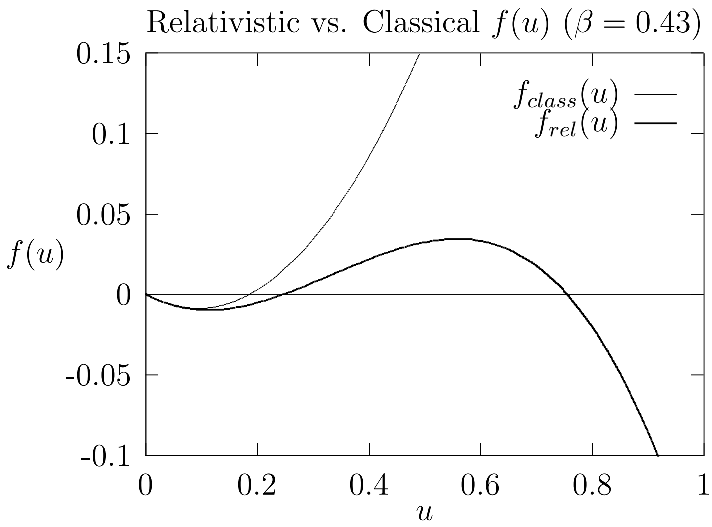


Figure 2: Here $\beta < 1/2$. There are orbits analogous to Kepler's ellipses, parabolas and hyperbolas.

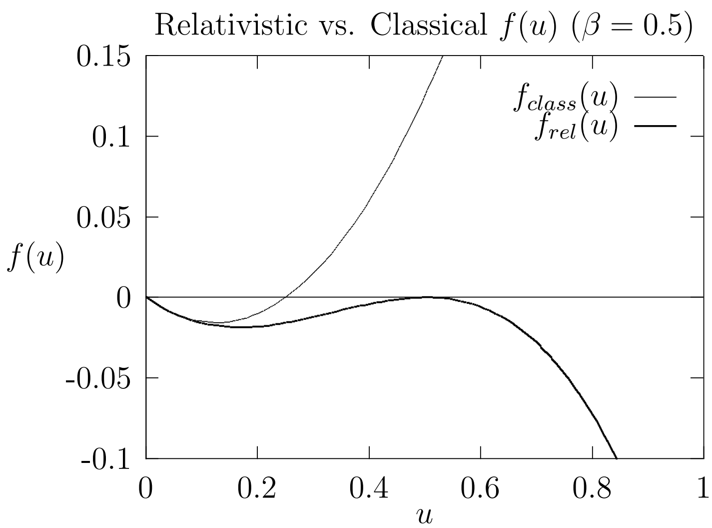


Figure 3: Here $\beta = 1/2$. All orbits with $\alpha > 0$ cross the horizon.

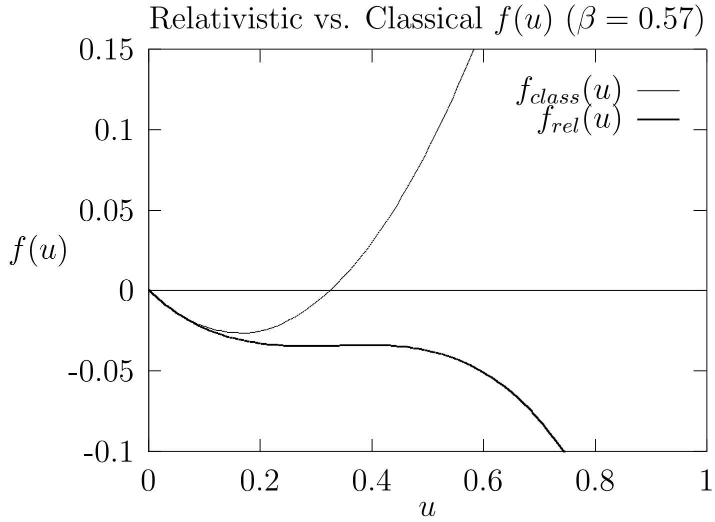


Figure 4: Here $\beta > 1/2$. All orbits cross the horizon.

i.e., $r = 6GM/c^2$. For $\beta^2 > 1/3$, all orbits extend to $u = 1$, i.e., to the event horizon.

The location of the local maximum in $f(u)$ is easily computed. The quadratic polynomial

$$f'(u) = 2u - \beta^2 - 3u^2$$

has roots at

$$u = \frac{1 \pm \sqrt{1 - 3\beta^2}}{3} .$$

Since $f''(u) = 2 - 6u$, the graph described by $f(u)$ curves upward for $u < 1/3$ and downward for $u > 1/3$. Thus, the local maximum in $f(u)$ occurs at

$$u = u_{\max} := \frac{1 + \sqrt{1 - 3\beta^2}}{3} .$$

Corresponding to

$$\alpha = f(u_{\max}) = \frac{1}{27}(1 + \sqrt{1 - 3\beta^2})^2(-1 + 2\sqrt{1 - 3\beta^2})$$

there is an unstable circular orbit at $u = u_{\max}$. For any given value of β, orbits with $\alpha > f(u_{\max})$ always reach the event horizon $u = 1$. If the angular momentum parameter β is chosen just right, one of the roots of $f'(u) = 0$ will coincide with one of the roots of $f(u) = 0$. In this case the latter equation has a double root and the zero energy ($\alpha = 0$) orbit will be circular. The requisite value of β is one such that $|\beta| = 1/2$, for which the circular orbit occurs at $u = 1/2$.[3] The unstable nature of this orbit is described mathematically through the observation that the differential equation

$$\left(\frac{du}{d\varphi}\right)^2 = u\left(\frac{1}{2} - u\right)^2$$

corresponding to $\alpha = 0$ and $\beta = 1/2$ has not only the solution $u = 1/2$, but also the solutions

$$u = \frac{1}{2}\tanh^2\left(\frac{\varphi}{2\sqrt{2}}\right)$$

and

$$u = \frac{1}{2}\coth^2\left(\frac{\varphi}{2\sqrt{2}}\right).$$

Each of these alternative solutions tends asymptotically to $u = 1/2$ as $\varphi \to \pm\infty$, while the first goes to $u = 0$ (i.e., $r = \infty$) and the second goes to $u = \infty$ (i.e., $r = 0$) at $\varphi = 0$. Hence, the slightest radial perturbation of the orbit will cause the object to spiral either outward to larger values of r or inward to smaller values of r.

By selecting an orbit such that α is sufficiently close to but slightly less than the value $\alpha = f(u_{\max})$ that corresponds to the local maximum of $f(u)$, one can obtain an orbit that is like a Kepler orbit at large distances from the black hole, i.e., an ellipse, parabola or hyperbola, but which loops around the black hole near $u_{\max}$ as many times as one might wish to do so. The calculation is particularly easy if one employs a "parabolic orbit" with

$$\alpha = 0\ ,\ \beta^2 = \frac{1}{4} - \epsilon^2\ ,$$

where $1 \gg \epsilon > 0$. This data corresponds to

$$E = 0, \text{ and } \ell = \pm\frac{4}{1 - 4\epsilon^2}.$$

Suppose $\varphi = 0$ when $u = 0$, i.e., when $r = \infty$. The number of times the black hole will be encircled by such an orbit is given by

$$N = \frac{1}{\pi}\int_0^{\frac{1}{2}-\epsilon} \frac{du'}{\sqrt{u'(\frac{1}{2} - \epsilon - u')(\frac{1}{2} + \epsilon - u')}}\ .$$

[3]Remember that $u = 1/2$ corresponds to $r = 4GM/c^2$.

Let us introduce a new integration variable x such that

$$u' = (\frac{1}{2} - \epsilon) \sin^2 (\frac{x}{2}) \ .$$

Then

$$\frac{du'}{\sqrt{u'(\frac{1}{2} - \epsilon - u')}} = dx \ ,$$

and

$$\sqrt{\frac{1}{2} + \epsilon - u'} = \sqrt{(\frac{1}{2} + \epsilon) - (\frac{1}{2} - \epsilon) \sin^2 (\frac{x}{2})} \ .$$

Thus,

$$N = \frac{1}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{(\frac{1}{2} + \epsilon) - (\frac{1}{2} - \epsilon) \sin^2 (\frac{x}{2})}} \ ,$$

or

$$N = \frac{1}{\pi\sqrt{1/2 + \epsilon}} K(k) \ ,$$

where

$$k^2 := \frac{1/2 - \epsilon}{1/2 + \epsilon}$$

and $K(k)$ is the complete elliptic integral that you will find in the tables of Jahnke and Emde. Values for certain specific values of k are given on p. 85. As $k \to 1$, $K(k) \to \infty$, while $K(0.9997015) = 5.0988$, which corresponds to $\epsilon = 0.000149$ and $N = 2.30$. On the other hand, $K(0.9999984769) = 7.7371$, which corresponds to $\epsilon = 0.0000007615$ and $N = 3.48$.

For smaller values of ϵ, i.e., k closer to 1, one can employ the approximation (see J & E, p. 73)

$$K(k) \approx -\frac{1}{2} \ln (\epsilon/4) \ ,$$

so, for such values of ϵ,

$$N(\epsilon) \approx -\frac{1}{\sqrt{2}\pi} \ln (\epsilon/4) \ .$$

Conversely

$$\epsilon(N) = 4e^{-\sqrt{2}\pi N} \ .$$

Thus, for example, if you wish to coast ten times about the black hole, you should arrange that $\epsilon \approx 2 \times 10^{-19}$.[4]

[4] In practice you would expect to have to make fine tuning course corrections as you approach $u = u_{\max}$.

You might also be interested in the time that it takes to complete one orbit about the black hole. Since

$$\frac{1}{2}r^2\dot{\varphi} = \frac{GM}{c\beta} ,$$

the area A enclosed by the orbit is related to the period P of the orbit by[5]

$$A = \frac{1}{2}\int r^2\, d\varphi = \left(\frac{GM}{c\beta}\right) P .$$

In particular, for a circular orbit, one obtains

$$cP = \beta \left[A/\left(\frac{GM}{c^2}\right)^2\right]\left(\frac{GM}{c^2}\right) .$$

In the case of the highly relativistic circular orbit at $r = 4GM/c^2$, where $\beta = 1/2$, we have

$$cP = 8\pi\frac{GM}{c^2} .$$

Thus, in this case, we arrive at the approximate formula

$$P = 1.5 \times 10^{-9} \text{ days } \times \left(\frac{M}{M_\odot}\right) ,$$

where $M_\odot$ is the mass of our Sun. This is the period as recorded by clocks on the orbiting space ship. The elapsed coordinate time t can be computed using

$$E = mc^2\left(1 - \frac{2GM}{c^2 r}\right)\dot{t} .$$

For an $\alpha = 0$ orbit at $r = 4GM/c^2$, $E = mc^2$ and

$$\dot{t} = 2 ,$$

i.e., the ratio of elapsed coordinate time to elapsed proper time is 2.

The voyage of the Stultitia Loquitur

The collapsed galaxy RS-232 has been the subject of intense study by inhabitants of the *Erasmus space station*, which has been in orbit at $r = 2000GM/c^2$

[5]This is, of course, a *coordinate area.*

about the central black hole. Unmanned vehicles have been sent toward the black hole, but only recently has space ship design improved enough to contemplate putting a human in orbit near $r = 4GM/c^2$. A zero-energy circular orbit at $r = 4GM/c^2$ is unstable. A slight perturbation outward and the spaceship will retreat from the black hole. A slight perturbation inward and the spaceship will plummet into the black hole.

In order to limit accelerations to something comparable to 1g, and to make the orbital period something like 1 year, one has to choose a black hole that is much more massive than the Sun. If one were to be orbiting a black hole of several solar masses at near light speed, 300,000 km/sec, in a circular orbit of radius 9 km, one would loop about the black hole some 1000 times a second. This is not a reasonable time scale for human participation. The centripetal acceleration would be 10^{10} km/sec^2, or 10^{12}g. Any maneuvering would require engines much more powerful than any likely ever to be available. Moreover, and more significantly, while you would not feel the acceleration, since you would be in free fall, you would be quite painfully aware of the gravitational gradients, i.e., the tidal forces, which neither your body nor the ship could survive.

The actual mass of the black hole about which Erasmus is orbiting is 10^8 solar masses. For such a black hole, it will be convenient to use

$$\frac{GM}{c^2} = 1.47 \times 10^8 \text{ km } \approx 1 \text{ a.u.}$$

as a unit of distance, and

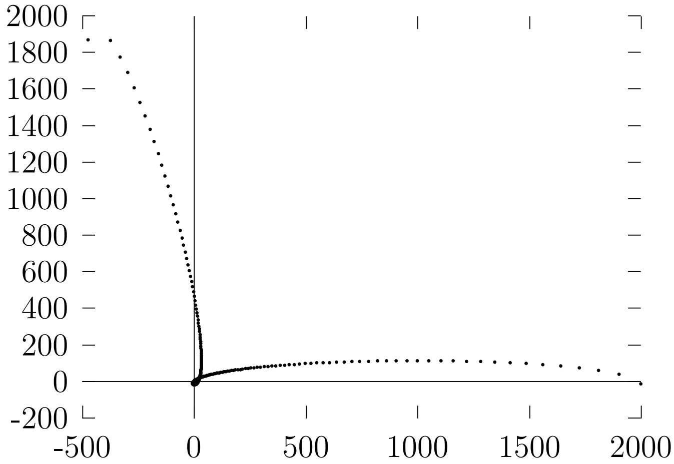
$$\frac{GM}{c^3} \approx 8.17 \text{ min}$$

as a unit of time. The space station is at $r = 2000GM/c^2$, i.e., approximately 2000 a.u. (or eleven light-days) from the black hole. Communications between your star ship and the Erasmus space station will not be rapid! Moreover, there should be interesting gravitational lens effects when a message is sent from a point located beyond the black hole. Kepler's third law

$$P = \frac{2\pi}{\sqrt{GM}} r^{3/2}$$

yields for the space station a period

$$P = 8.74 \times 10^{-8} \text{ years } \times \left(\frac{M}{M_\odot}\right) \ ,$$



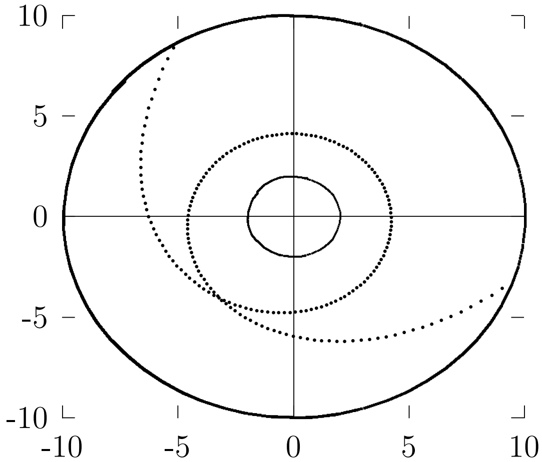


Figure 5: The orbit of the Stultitia Loquitur from its departure from the space station Erasmus until its return three years later, together with a close up view of the orbit of the Stultitia Loquitur as it encircles the Schwarzschild black hole.

that is, about 8 years 9 months. There is not much difference between proper time and coordinate time on the Erasmus space station, i.e., $\dot{t} \approx 1$.

We have seen that the proper time required to fall to $r \approx 0$ from the orbit of Erasmus is approximately one-fifth of the orbital period of Erasmus, i.e., about a year and one-half. This should be about half the proper time required for the trip that you contemplate, not including the time you spend looping about the black hole near $r = 4GM/c^2$. Except for the time you spend in orbit near $r = 4GM/c^2$, when $\dot{t} \approx 2$, you expect that for most of the trip $\dot{t}$ will not be much larger than 1. During each orbit you execute about the black hole, a proper time of thirty-six hours will elapse, i.e., seventy-two hours of coordinate time (which is approximately the proper time as measured on the Erasmus space station).

Now, you will not be traveling in a zero energy orbit. Rather, your orbit will be a precessing elliptical orbit (with $E_{\text{class}} < 0$). At perihelion and aphelion, $\dot{r} = 0$. Therefore, Eq. (9.1) tells us that

$$\begin{aligned} \left(\frac{E}{mc^2}\right)^2 &= \left(1 - \frac{2GM/c^2}{r_p}\right)\left(1 + \frac{(\ell/mc)^2}{r_p^2}\right) \\ &= \left(1 - \frac{2GM/c^2}{r_a}\right)\left(1 + \frac{(\ell/mc)^2}{r_a^2}\right), \end{aligned}$$

where we are interested in the case $r_a = 2000GM/c^2$ and $r_p \approx 4GM/c^2$. Using this pair of equations we can evaluate the constants of the motion E and ℓ, which turn out to be very close to

$$E/mc^2 = 0.9995, \quad \ell/mc = \pm 4.000GM/c^2.$$

Conversely, the equation

$$(0.9995)^2 r^3 = (r - 2)(r^2 + 16)$$

has three solutions, one big one and two small ones:

$$r = 1992.475844, \quad r = 4.191886834, \quad r = 3.832268567.$$

The first corresponds to the aphelion and the second to the perihelion of your intended orbit. The third is the aphelion of an orbit that plunges into the black hole. You will want to be very careful not to move onto that orbit as you reach perihelion at $r \approx 4.1919$.

Ex. 20 *Show that ℓ^2 can be determined using the formula*

$$\ell^2 = \frac{2GM/c^2}{\left(\frac{1}{r_p} + \frac{1}{r_a}\right) - (2GM/c^2)\left(\frac{1}{r_p^2} + \frac{1}{r_p r_a} + \frac{1}{r_a^2}\right)},$$

which is a straightforward generalization of a Newtonian result.

Using the formula $\ell = mr^2\dot{\varphi}$ and the value $\ell/mc = 4.0GM/c^2$, we obtain

$$\dot{\varphi} = 1.0 \times 10^{-6} \text{ at aphelion, and } \dot{\varphi} \approx 0.25 \text{ at perihelion.}$$

Ex. 21 *How rapidly will you actually be moving at perihelion?*

The orbital speed of the Erasmus space station, in a circular orbit of radius $r = 2000GM/c^2$, is approximately $0.022c$. Using the engines of your space ship, you will reduce your speed from $0.022c$ to $0.002c$, which corresponds to $\dot{\varphi} \approx 10^{-6}$ (in units of radians per 8.17 minutes). You can accomplish such a change in speed in one week by accelerating at a mere 1g. An "elliptical" orbit (actually a rosette) will take you down to the vicinity of $r = 4GM/c^2$, where you will execute almost two laps before returning to the orbit of the Erasmus space station. The time that elapses from one aphelion to the next will be approximately three years, during which time the Erasmus space station will have traversed about one-third of its orbit.

Interestingly, the relativistic precession of your elliptical orbit will bring you back very close to the Erasmus space station three years after your departure. Incidentally, three years is approximately 200,000 times 8.17 minutes. Actually, your first aphelion occurs after a proper time $\tau = 200,940$ elapses, and that corresponds to $t = 201,234$. During this time the angle φ has gone from zero to $\varphi = 14.36$ radians, which is 2.29 revolutions. Now, 0.29 revolution is equal to 102.8 degrees. This can be clearly seen in the accompanying overall picture of the orbit. The close up view shows why 2.29 and not 1.29 revolutions have taken place.

Now, noting that, for the circular orbit of the Erasmus space station, $\ell = 44.75$ or $\dot{\varphi} = 11.2 \times 10^{-6}$, you can compute where the space station will be at $t = 201,234$. You just need to know that $\dot{t} = 1.00125$, so at $t = 201,234$ the elapsed proper time on the space station is $\tau = 200,983$. At this time the space station will be located at $\varphi = 129$ degrees, a mere 26 degrees (i.e., about 76 light-hours or 8.22×10^{10} km) ahead of where the Stultitia Loquitur will be.

Ex. 22 *Assuming that you accelerate at 1g for half the time, and decelerate at 1g for the remaining time, compute the time it will it take you to catch up with the Erasmus space station, after you reach aphelion.*[6]

[6] Note that your clocks will be several hours behind the clocks on the Erasmus space station, when you finally return there.

The engines of your new ship, the *Stultitia Loquitur*, are the most reliable that are currently available. Instruments will constantly monitor your position, and on-board computers will calculate and display your coordinates r, θ, φ, t, the first and second derivatives of these coordinates with respect to the proper time τ, and the orbital parameters E and ℓ.

Ex. 23 *Describe measurements that might allow you to determine your actual position.*

You are notified that the ship is ready to leave. You climb aboard and begin your epic journey. Bon voyage!

Ex. 24

(a) Describe the appearance of the Stultitia Loquitur as seen from the Erasmus space station as the former orbits at $r = 4GM/c^2$.

(b) Describe the appearance of the Erasmus space station as seen from the Stultitia Loquitur as the latter orbits at $r = 4GM/c^2$.

In the case of Schwarzschild spacetime, at least outside the event horizon, one can arbitrarily designate the Schwarzschild time t as a sort of *Greenwich time*, in terms of which one can say whether or not two spatially separated events are to be regarded as happening "simultaneously." Thus, for example, we can determine the position of the Stultitia Loquitur and the Erasmus space station in the Schwarzschild chart at a particular time t. However, in a spacetime that corresponds to a truly dynamical solution of the Einstein field equations there exists no such obvious choice of a "universal time."