Chapter 8

Spacetime Curvature

Flat Spacetime

Before we consider true curved spacetime, it may be useful to look at special relativity from the perspective of differential geometry, using non-Cartesian coordinates to describe a local region of flat spacetime.

The use of non-inertial frames of reference and the use of non-Cartesian coordinates is common in classical mechanics. Rigid body motions are frequently described in terms of a basis of unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ which are fixed in the body and which correspond to the principal axes of the moment of inertia tensor of the body. Such an analysis leads, for example, to the Euler equations of motion of a rigid body.

In terms of moving frames of reference, there may appear centrifugal and Coriolis forces, i.e., forces of purely inertial origin, as when the motion of projectiles fired from the earth's surface is described in terms of a frame of reference which is fixed with respect to the earth's surface. In this case, the moving vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are time-dependent, and that time dependence must be taken into account.

If curvilinear coordinates are used in classical mechanics, one must bear in mind that the associated unit vectors, e.g., $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$ in the case of spherical polar coordinates, are no longer independent of location in space. This spatial dependence of the basic unit vectors must be taken into account.

A similar situation arises in spacetime when a tetrad is used which depends upon one or more spacetime coordinates. In the case of flat spacetime, there will be a spacetime dependence of the tetrad if a non-Cartesian frame is employed. In the case of curved spacetime, there will always be a spacetime dependence of the tetrad.

Suppose we wish to introduce in flat spacetime an "orthonormal tetrad" that is spacetime dependent. At each spacetime point we can relate the "moving frame" $\mathbf{e}_a(\mathbf{x}), (a = 1, 2, 3, 4)$ to a Cartesian reference frame $\mathbf{e}'_b, (b = 1, 2, 3, 4)$ extending throughout the flat spacetime. Then

$$\mathbf{e}_a(\mathbf{x}) = \Lambda_a{}^b(\mathbf{x})\mathbf{e}'_b \; ,$$

where the Lorentz transformation matrix $\Lambda_a{}^b(\mathbf{x})$ differs from point to point in spacetime.

Let's consider what happens when one applies the differential operator d to $\mathbf{e}_a$, assuming $d\mathbf{e}'_b = 0$. We obviously get

$$d\mathbf{e}_a(\mathbf{x}) = (d\Lambda_a{}^b(\mathbf{x}))\mathbf{e}'_b \; .$$

If one applies d again, one gets

$$d^2\mathbf{e}_a(\mathbf{x}) = (d^2\Lambda_a{}^b(\mathbf{x}))\mathbf{e}'_b = 0 \; .$$

It is easy enough to solve for $\mathbf{e}'_b$ in terms of $\mathbf{e}_c(\mathbf{x})$, and thus show that

$$d\mathbf{e}_a(\mathbf{x}) = [\Lambda^c{}_b(\mathbf{x})d\Lambda_a{}^b(\mathbf{x})]\mathbf{e}_c(\mathbf{x}) =: \Gamma_a{}^c(\mathbf{x})\mathbf{e}_c(\mathbf{x}) \; .$$

The matrix $\Gamma_a{}^c$ of 1-forms can be expressed in the form

$$\Gamma_a{}^c = e^b \, \Gamma_{ba}{}^c \; ,$$

where the quantities $\Gamma_{ba}{}^c$ are the components of the connection relative to the orthonormal basis $\{e^a\}$. From the fact that the Lorentz transformation matrix $\Lambda_a{}^b(\mathbf{x})$ satisfies the relation

$$\Lambda^c{}_b(\mathbf{x})\Lambda_a{}^b(\mathbf{x}) = \delta^c_a$$

one may infer that the matrix $\Gamma_{ab}(\mathbf{x})$ is skew symmetric. The same observation follows more directly from the fact that we are employing an orthonormal tetrad, for which the metric tensor g_{ab} is spacetime independent, i.e., $dg_{ab} = 0$. Thus, when an orthonormal basis is employed, Γ_{ab} is equal to the Ricci rotation matrix Ω_{ab} with respect to that basis.

Applying the d operator to the equation $d\mathbf{e}_a = \Gamma_a{}^b\mathbf{e}_b$, we obtain the very important result

$$\begin{aligned} d^2\mathbf{e}_a(\mathbf{x}) &= d[\Gamma_a{}^b(\mathbf{x})\mathbf{e}_b(\mathbf{x})] \\ &= [d\Gamma_a{}^b(\mathbf{x})]\mathbf{e}_b(\mathbf{x}) - \Gamma_a{}^b(\mathbf{x})d\mathbf{e}_b(\mathbf{x}) \\ &= [d\Gamma_a{}^c(\mathbf{x}) - \Gamma_a{}^b(\mathbf{x})\Gamma_b{}^c(\mathbf{x})]\mathbf{e}_c(\mathbf{x}) \; . \end{aligned}$$

Thus, in flat spacetime,

$$d\Gamma_a{}^c - \Gamma_a{}^b\Gamma_b{}^c = 0 \ . \tag{8.1}$$

As a matter of fact, this is a sufficient as well as a necessary condition for the spacetime to be flat! It can be shown that whenever this object, called the curvature 2-form, vanishes, it is possible to introduce a Cartesian coordinate system and an associated uniform orthonormal tetrad $\{\mathbf{e}'_a\}$.

As an exceedingly simple illustration of these ideas, consider a flat spacetime in which cylindrical coordinates ρ, ϕ, z, t are employed, together with an orthonormal basis $\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z, \mathbf{e}_t$ adapted to such coordinates. In this case we may write

$$\begin{aligned}
\mathbf{e}_\rho &= \cos\phi\mathbf{e}'_1 + \sin\phi\mathbf{e}'_2 \ , \\
\mathbf{e}_\phi &= -\sin\phi\mathbf{e}'_1 + \cos\phi\mathbf{e}'_2 \ , \\
\mathbf{e}_z &= \mathbf{e}'_3 \ , \\
\mathbf{e}_t &= \mathbf{e}'_4 \ .
\end{aligned}$$

Here the basis $\{\mathbf{e}'_a\}$ is spacetime independent, so $d\mathbf{e}'_a = 0$. Hence,

$$\begin{aligned}
d\mathbf{e}_\rho &= -\sin\phi \ d\phi \ \mathbf{e}'_1 + \cos\phi \ d\phi \ \mathbf{e}'_2 \\
&= -\sin\phi \ d\phi[\cos\phi\mathbf{e}_\rho - \sin\phi\mathbf{e}_\phi] \\
&\quad + \cos\phi \ d\phi[\sin\phi\mathbf{e}_\rho + \cos\phi\mathbf{e}_\phi] \\
&= d\phi \ \mathbf{e}_\phi \ , \\
d\mathbf{e}_\phi &= -\cos\phi \ d\phi \ \mathbf{e}'_1 - \sin\phi \ d\phi \ \mathbf{e}'_2 \\
&= -\cos\phi \ d\phi[\cos\phi\mathbf{e}_\rho - \sin\phi\mathbf{e}_\phi] \\
&\quad - \sin\phi \ d\phi[\sin\phi\mathbf{e}_\rho + \cos\phi\mathbf{e}_\phi] \\
&= -d\phi \ \mathbf{e}_\rho \ , \\
d\mathbf{e}_z &= 0 \ , \\
d\mathbf{e}_t &= 0 \ .
\end{aligned}$$

Recall that in general $d\mathbf{e}_b = \Gamma_b{}^c\mathbf{e}_c$, and, since the basis $\{\mathbf{e}_a\}$ is orthonormal, $\Gamma_{bc} = \Omega_{bc}$, the latter being a skew symmetrical matrix of one-forms. It is quite apparent that in the present problem all matrix elements of Ω_{bc} vanish except for $\Omega_{12} = -\Omega_{21} = d\phi$. It is also obvious that Eq. (8.1) is satisfied.

Ex. 14 *Evaluate the Ricci rotation matrix for flat spacetime using spherical polar coordinates r, θ, ϕ, t and a tetrad adapted to these coordinates. Then show that Eq. (8.1) is satisfied.*

Curved Spacetime

Let's no longer assume that one can introduce a tetrad $\mathbf{e}'_a$ such that $d\mathbf{e}'_a = 0$. In this case, no matter what tetrad $\mathbf{e}_a$ we use, we shall get $d\mathbf{e}_a \neq 0$ and $d^2\mathbf{e}_a \neq 0$. Although

$$d\mathbf{e}_a = d(\Lambda_a{}^b\mathbf{e}'_b) = (d\Lambda_a{}^b)\mathbf{e}'_b + \Lambda_a{}^b d\mathbf{e}'_b$$

does *not* transform as a four-vector under Lorentz transformations, nevertheless $d^2\mathbf{e}_a$ does transform as a four-vector, for

$$d^2\mathbf{e}_a = (d^2\Lambda_a{}^b)\mathbf{e}'_b - (d\Lambda_a{}^b)d\mathbf{e}'_b + (d\Lambda_a{}^b)d\mathbf{e}'_b + \Lambda_a{}^b d^2\mathbf{e}'_b = \Lambda_a{}^b d^2\mathbf{e}'_b\ .$$

The components of this object, which is a $(2, 1)$-tensor, are given by

$$d^2\mathbf{e}_a = \{d\Gamma_a{}^c - \Gamma_a{}^b\Gamma_b{}^c\}\mathbf{e}_c\ .$$

Since this object transforms the same way as $\mathbf{e}_a$, the object

$$\mathbb{R} := \frac{1}{2}\mathbf{e}^a \wedge d^2\mathbf{e}_a = \frac{1}{2}\{d\Gamma_a{}^c - \Gamma_a{}^b\Gamma_b{}^c\}\mathbf{e}^a \wedge \mathbf{e}_c \tag{8.2}$$

is an *invariant* geometrical object that is a $(2, 2)$-tensor (having the character of a 2-form and a bivector simultaneously). It is this object that characterizes the *curvature* of spacetime. In particular, $\mathbb{R} = 0$ is the necessary and sufficient condition for spacetime to be flat. One may, of course, express $\mathbb{R}$ in terms of components R_{abcd} relative to any given basis, i.e.,

$$\mathbb{R} = \frac{1}{4}e^a e^b R_{ab}{}^{cd}\mathbf{e}_c \wedge \mathbf{e}_d$$

or, equivalently,

$$d\Gamma_c{}^d - \Gamma_c{}^b\Gamma_b{}^d = \frac{1}{2}e^a e^b R_{abc}{}^d.$$

Recall that whenever we write a differential form to the right of a tangent vector field, the differential form is to be regarded as a linear functional acting leftward upon the tangent vector field. Therefore, we may regard the Riemann tensor $\mathbb{R}$ to be a *linear operator* on differential 2-forms. A second example of a linear operator on differential 2-forms is the unit $(2, 2)$-tensor

$$\mathbb{I}^{(2)} := \frac{1}{2}\mathbb{I} \wedge \mathbb{I} = \frac{1}{2}e^a e^b\ \mathbf{e}_a \wedge \mathbf{e}_b\ ,$$

and still another is the *duality operator*

$$\mathbb{D} := \frac{1}{4}\epsilon_{abcd}e^a e^b\ \mathbf{e}^c \wedge \mathbf{e}^d\ ,$$

where the Levi-Civita permutation symbol is defined by

$$\epsilon_{abcd} := \begin{cases} 1 \text{ if } a,b,c,d \text{ is a cyclic permutation of } 1,2,3,4 \\ -1 \text{ if } a,b,c,d \text{ is an anti-cyclic permutation of } 1,2,3,4 \\ 0 \text{ otherwise} \end{cases}$$

Because

$$\mathbb{D}^2 = -\mathbb{I}^{(2)},$$

the duality operator $\mathbb{D}$ separates the six-dimensional space of 2-forms into two three-dimensional subspaces, a subspace of two-forms W for which

$$\mathbb{D}W = +iW,$$

and a subspace of two-forms W for which

$$\mathbb{D}W = -iW.$$

We may introduce projection operators,

$$\begin{aligned} \mathbb{P} &:= \frac{1}{2}(\mathbb{I}^{(2)} - i\mathbb{D}) , \\ \mathbb{P}^* &:= \frac{1}{2}(\mathbb{I}^{(2)} + i\mathbb{D}) , \end{aligned}$$

which project two-forms onto these two three-dimensional subspaces. Two-forms W which satisfy the relation $\mathbb{P}W = W$ will be called *self-dual* 2-forms, while 2-forms W which satisfy the relation $\mathbb{P}^*W = W$ will be called *anti-self-dual* 2-forms.[1] If one is using an orthonormal basis e^a $(a = 1,2,3,4)$ for one-forms, it is easy to demonstrate that $\mathbb{D}e^2e^3 = e^1e^4$ and $\mathbb{D}e^1e^4 = -e^2e^3$, etc., so

$$\begin{cases} e^2e^3 - ie^1e^4 \ , \\ e^3e^1 - ie^2e^4 \ , \\ e^1e^2 - ie^3e^4 \ , \end{cases}$$

are all eigen-two-forms of $\mathbb{D}$ corresponding to eigenvalue $+i$, and the complex conjugates of these 2-forms are eigen-two-forms of $\mathbb{D}$ corresponding to eigenvalue $-i$. Of course, any three linearly independent linear combinations of the three basic 2-forms would suffice.

The Riemann curvature tensor $\mathbb{R}$ can be invariantly decomposed into several parts by writing out

$$\mathbb{R} = (\mathbb{P} + \mathbb{P}^*)\mathbb{R}(\mathbb{P} + \mathbb{P}^*) \ .$$

[1] Note that some people employ a slightly different definition of the duality operator so that the eigenvalues are ± 1 instead of $\pm i$. Our duality operator itself is real.

In this way we obtain

$$\mathbb{R} = \mathbb{C} + \mathbb{E} + \frac{R}{12}\mathbb{I}^{(2)} ,$$

where R is called the Ricci Scalar and the symmetric trace-free operators[2]

$$\begin{aligned} \mathbb{E} &:= \mathbb{P}\mathbb{R}\mathbb{P}^* + \mathbb{P}^*\mathbb{R}\mathbb{P} , \\ \mathbb{C} &:= \mathbb{P}\mathbb{R}\mathbb{P} + \mathbb{P}^*\mathbb{R}\mathbb{P}^* - \frac{R}{12}\mathbb{I}^{(2)} , \end{aligned}$$

are called the *Einstein part* and the *Weyl conform part* of the Riemann curvature tensor, respectively.

The Grassmann inner product is a generalization of the ordinary scalar (or dot) product. The product $u \ulcorner v$ vanishes whenever the degree p of u is greater than the degree q of v. When $p \leq q$, $u \ulcorner v$ is that form of degree $q - p$ that has the property

$$w \cdot (u \ulcorner v) = (wu) \ulcorner v$$

for all forms w of degree $q - p$. This $\ulcorner$ product satisfies

$$w \ulcorner (v \ulcorner u) = (wv) \ulcorner u$$

for all u, v, w regardless of their degrees, and

$$w \ulcorner (vu) = v(w \ulcorner u) - (w \ulcorner v)u$$

for all 1-forms u, w and p-forms v.

Similarly, the Grassmann inner product $\mathbf{u} \urcorner \mathbf{v}$ vanishes whenever the degree p of $\mathbf{u}$ is less than the degree q of $\mathbf{v}$. When $p \geq q$, $\mathbf{u} \urcorner \mathbf{v}$ is that vector of degree $p - q$ that has the property

$$(\mathbf{u} \urcorner \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w})$$

for all vectors w of degree $p - q$. This $\urcorner$ product satisfies

$$(\mathbf{u} \urcorner \mathbf{v}) \urcorner \mathbf{w} = \mathbf{u} \urcorner (\mathbf{v} \wedge \mathbf{w})$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ regardless of their degrees, and

$$(\mathbf{u} \wedge \mathbf{v}) \urcorner \mathbf{w} = (\mathbf{u} \urcorner \mathbf{w})\mathbf{v} - \mathbf{u}(\mathbf{v} \urcorner \mathbf{w})$$

for all 1-vectors $\mathbf{u}, \mathbf{w}$ and p-vectors $\mathbf{v}$.

[2] We have not yet demonstrated that either $\mathbb{E}$ or $\mathbb{C}$ is symmetric or trace-free.

It turns out that

$$\mathbb{E} = -\frac{1}{4}\left(\mathbb{S} \wedge \mathbb{I} + \mathbb{I} \wedge \mathbb{S}\right) \ ,$$

where the components of the $(1,1)$-tensor

$$\mathbb{S} := -e^a \ulcorner \mathbb{E} \urcorner \mathbf{e}_a$$

are nothing but the components of the trace-free part of the Ricci tensor

$$e^a R_{ab} \mathbf{e}^b := -e^a \ulcorner \mathbb{R} \urcorner \mathbf{e}_a \ .$$

The components of $\mathbb{C}$ have nothing to do with the Ricci tensor.

In any region of spacetime that is unoccupied by sources of gravity Einstein postulated[3] the field equations

$$\mathbb{E} = 0, \quad R = 0,$$

or, equivalently,

$$R_{ab} = 0.$$

In the case of a vacuum spacetime, any curvature is reflected in a nonvanishing $\mathbb{C}$. Because of the symmetry of the Weyl conform tensor $\mathbb{C}$, information contained in the term $\mathbb{P}^*\mathbb{R}\mathbb{P}^* - \frac{R}{12}\mathbb{P}^*$ is duplicated in the term $\mathbb{P}\mathbb{R}\mathbb{P} - \frac{R}{12}\mathbb{P}$. It suffices, therefore, to work with the simpler symmetric trace-free $(2,2)$-tensor

$$\mathbb{C}_+ := \mathbb{P}\mathbb{R}\mathbb{P} - \frac{R}{12}\mathbb{P} \ .$$

Generally speaking, we shall suppress the subscript $+$ whenever no confusion is apt to arise.

Evaluation of the Curvature Tensor

It is important to recognize that the basis of tangent vectors and one-forms employed in the definition of $\mathbb{R}$ can be *any* basis, and need not be an orthonormal basis. With respect to any given basis the components of the abstract geometrical object $\mathbb{R}$ can be defined by

$$\mathbb{R} = \frac{1}{4} R_{ab}{}^{cd} \, e^a e^b \, \mathbf{e}_c \wedge \mathbf{e}_d \ .$$

[3]At one point Einstein considered requiring only $\mathbb{E} = 0$ with R merely a constant, but he later regretted ever having considered the introduction of such a *cosmological constant.*

If one wishes, one may think of the the object $R_{ab}{}^{cd}$ as a 6×6 matrix, the rows and columns of which are labeled by *pairs* of indices.

We have already seen how the connection one-form $\Gamma_b{}^c$ can be evaluated. In particular, when a natural basis $\boldsymbol{\partial}_b$ was employed, we found that

$$\Gamma_b{}^c = dx^a \Gamma_{ab}{}^c,$$

where the Christoffel symbol $\Gamma_{ab}{}^c$ was given by

$$\Gamma_{ab}{}^c = \frac{1}{2} g^{cd} \left[\frac{\partial g_{ad}}{\partial x^b} + \frac{\partial g_{db}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^d} \right].$$

If one evaluates

$$d\Gamma_a{}^c - \Gamma_a{}^b \Gamma_b{}^c = \frac{1}{2} e^m e^n R_{mna}{}^c$$

while continuing to employ a natural basis $\boldsymbol{\partial}_b$, one finds that R_{abcd} can be expressed in the form

$$\begin{aligned} R_{abcd} \quad = \quad & -\frac{1}{2} \left\{ \frac{\partial^2 g_{ad}}{\partial x^c \partial x^b} - \frac{\partial^2 g_{bd}}{\partial x^c \partial x^a} - \frac{\partial^2 g_{ac}}{\partial x^d \partial x^b} + \frac{\partial^2 g_{bc}}{\partial x^d \partial x^a} \right\} \\ & + g_{mn} \left\{ \Gamma_{ca}{}^m \Gamma_{db}{}^n - \Gamma_{da}{}^m \Gamma_{cb}{}^n \right\}. \end{aligned}$$

Ex. 15 *Verify the above expression for R_{abcd}. Moreover, show that*

(a) $R_{cdab} = R_{abcd}$,

(b) $R_{abcd} + R_{acdb} + R_{adbc} = 0$.

The Riemann tensor can, therefore, be thought of as a 6×6 symmetric matrix, the rows and columns of which are labeled by index pairs. Thus, in four dimensions the total number of independent components of the Riemann curvature tensor turns out to be twenty.

For many years almost all calculations of the curvature tensor involved component by component evaluation of the Christoffel symbols and the components R_{abcd} relative to a natural basis. Although the so-called *vier-bein*[4] technology had already existed for quite a long time, it was only in the 1960's that the use of orthonormal and null tetrads[5] began to dominate in practical calculations.

[4] This German term was used for many years.

[5] The spinorial formalism is equivalent to the null tetrad formalism.