

# Chapter 7

## Curvilinear Coordinates

Even in connection with special relativity it is convenient to be able to use non-Cartesian coordinates. For example, no one would try to calculate the electromagnetic fields associated with radiation from a dipole antenna using Cartesian coordinates. Invariably spherical polar coordinates would be used for this type of problem. When we pass over to general relativity, in which spacetime is no longer flat, there will be no Cartesian coordinates, and we shall be obliged to use more general curvilinear coordinates to describe spacetime events.

Learning to cope with the notion of curved spacetime is not a trivial endeavor, especially when one considers the fact that a curved four dimensional space cannot necessarily be portrayed as a hypersurface in a five dimensional flat space. In general the flat space in which you would have to embed a curved spacetime would be of much higher dimensionality. For this reason, one must learn to understand curvature without being able to look at it from “outside.”

This objective might seem to be well beyond reach of mere mortals. Nevertheless, one might imagine an intelligent two dimensional being actually discovering the curvature of the earth upon which he lives, in spite of the fact that he cannot enter the third (vertical) dimension. He might know, for example, the distances between a number of cities on the earth’s surface, i.e., the distances along the straightest possible paths between the cities. Chances are that, once he considered four such cities, discrepancies with two dimensional Euclidean geometry would be manifest, at least if the distances involved were substantial. Notice that his being able to infer the non-flatness of the earth in no way depends upon his attempting to use Cartesian coordinates; it just has to do with distances between points. The points can be labeled in whatever way one pleases. Names such as Los Angeles, New York,

Rio de Janeiro, etc. will do just fine, as long as there aren't too many. If there are a lot of points, it might be advantageous to distinguish the points by giving numerical labels to them. Labels consisting of pairs of real numbers would be the best idea, since then we could arrange that points which are "near" one another could be issued labels which are "near" one another numerically.

All this presupposes a notion of straightness, and a notion of distance. These are essential ingredients of the spaces with which we shall deal. Implicit also is the assumption that the spaces are not too "rough." In the neighborhood of any point, it is assumed that the space resembles a flat space, in which the laws of Euclidean geometry hold.

Let's consider a simpler two dimensional space, say the surface of the earth. If we consider a particular point on the earth's surface, nothing prevents us from attributing to each geographical point a pair of coordinates (e.g., distance North and distance East of a pair of intersecting reference lines), providing we restrict attention to a small enough neighborhood of the chosen point. Quite adequate maps of New York State can be made using such coordinates. Similarly, maps of the other states can be constructed in the same way, and we can arrange it that the maps overlap a little. A careful comparison of the overlap region as portrayed on the New York and Vermont maps might reveal slight discrepancies that are not very obvious when one looks at one map or the other. Neither the New York nor the Vermont map would give the "true" picture of the overlap region, which can only be conveyed on a globe. Here we have a situation in which the same region of the earth's surface is being described in two coordinate systems. The New York map uses a coordinate system, with coordinates  $(x, y)$ , and the Vermont map uses a coordinate system, with coordinates  $(x', y')$ , describing the very same point near the border of New York and Vermont.

These maps, for which we shall prefer the nautical term "chart," are really ordered pairs  $(\mathcal{U}, \phi)$ , where  $\mathcal{U}$  is a subset of the set  $\mathcal{M}$  of all points on the earth's surface, and where  $\phi$  is what mathematicians call a one-one *map* from  $\mathcal{U}$  to open sets in  $R^2$ , i.e., from the set  $\mathcal{U}$  to ordered pairs of real numbers (the coordinates). Since one chart fails to cover all points on the globe, we also introduce the notion of an "atlas," i.e., a collection of charts that together cover all points on the globe, and which overlap one another as did our charts of New York and Vermont.

It is not my intention to provide the rigorous definitions and rigorous chain of argument that the reader might find in a book such as *The large scale structure of spacetime* by Hawking and Ellis. Rather, by providing an intuitive understanding of the concepts, I seek to make it a little easier for

the reader who has limited experience with the ways of modern mathematics to cope with the more demanding sections of such books.<sup>1</sup>

## Differentiable Manifolds

There are two important concepts involved in the mathematical model of spacetime; one is the concept of a *differentiable manifold*  $\mathcal{M}$ , and the other is the concept of a Lorentz *metric*  $\mathcal{G}$ . Relativists tend always to behave as if all functions with which they deal are infinitely differentiable, i.e.,  $C^\infty$  functions, even though less differentiable functions do play a conspicuous role, for example, in connection with the subject of colliding plane gravitational waves. Here we shall gloss over the actual degree of differentiability demanded.

An  $n$ -dimensional *differentiable manifold* is a set  $\mathcal{M}$  of points together with an *atlas*  $\{\mathcal{U}_\alpha, \phi_\alpha\}$ . Here the various overlapping  $n$ -dimensional charts are distinguished by the subscript  $\alpha$ . The sets  $\{\mathcal{U}\}$  of the atlas must cover  $\mathcal{M}$ , i.e., the set  $\mathcal{M}$  is the union of all the sets  $\mathcal{U}_\alpha$ , or

$$\mathcal{M} = \cup_\alpha \mathcal{U}_\alpha.$$

Moreover, if the set intersection  $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$  is non-empty, then the map

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \rightarrow \phi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

must be a differentiable map of an open subset of  $R^n$  to an open subset of  $R^n$ . Here  $\phi_\beta^{-1}$  is the inverse map to the map  $\phi_\beta$  which maps  $\mathcal{U}_\beta$  to  $R^n$ , and  $\phi_\alpha \circ \phi_\beta^{-1}$  is the *composition* of two maps. By *composition* one means that the map on the right has as its range a subset of the domain of the function of the left, so it makes sense to apply the map on the left to the result of applying the map on the right.

For one and the same manifold  $\mathcal{M}$  one can have different atlases, i.e., sets of charts that suffice to cover  $\mathcal{M}$ . It is useful to introduce the notion of a *complete atlas*, which is the set of all possible coordinate systems covering  $\mathcal{M}$ . This permits one to introduce the concept of *topology*, i.e., open sets, in the manifold  $\mathcal{M}$  itself. The *open sets* of  $\mathcal{M}$  are defined to be all sets which are expressible as unions of the  $\mathcal{U}_\alpha$  belonging to the complete atlas.

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<sup>1</sup>Even if you find the early chapters of Hawking and Ellis to be impenetrable, do read the rest of the book, starting with chapter five on exact solutions of the Einstein field equations.

## Tangent Manifold at a Point

We have an intuitive notion that in a sufficiently small neighborhood of a point, any curved space can be approximated by a flat space. In the case of the 2-dimensional surface of the earth, there is a 2-dimensional plane tangent to the earth's surface at any selected point. The same notion can be extended to four or more generally to  $n$  dimensions. Our objective is to make this idea of a *tangent manifold* more rigorous.

We can begin by considering a family of *curves* through the selected point  $p$ . In a parametric representation of such a curve, we have a function  $\lambda(t) : R^1 \rightarrow \mathcal{M}$ . Of course, we shall restrict the curves considered by imposing certain smoothness criteria upon the function  $\lambda(t)$ . The parameter value  $t = 0$  will correspond to the chosen point in the manifold, through which all the curves considered will pass, i.e.,  $\lambda(0) = p$ .

For any given selection of the function  $\lambda$ , one can consider the *directional derivative* of  $f : \mathcal{M} \rightarrow R^n$ . Explicitly, such a directional derivative is given by

$$\left(\frac{\partial f}{\partial t}\right)_\lambda \Big|_{t=0} = \lim_{t \rightarrow 0} \frac{1}{t} \{f(\lambda(t)) - f(\lambda(0))\} .$$

We may regard the corresponding *tangent vector*  $\mathbf{X}_\lambda|_p$  to be a linear functional on the space of differentiable test functions  $f$ . This is the linear functional that maps the function  $f$  at  $p$  into the number  $(\partial f/\partial t)_\lambda|_{t=0}$ .

This can be made more explicit by introducing local coordinates in a neighborhood of the point  $p$ . If one uses a chart in which the local coordinates are denoted by  $x^1, x^2, \dots, x^n$ , then

$$\left(\frac{\partial f}{\partial t}\right)_\lambda \Big|_{t=0} = \sum_{j=1}^n \frac{dx^j(\lambda(t))}{dt} \Big|_{t=0} \frac{\partial f}{\partial x^j} \Big|_{\lambda(0)} = \frac{dx^j(\lambda(t))}{dt} \frac{\partial f}{\partial x^j} \Big|_{\lambda(0)} ,$$

where the summation convention is being employed. Thus, every tangent vector at  $p$  can be expressed as a linear combination of those tangent vectors  $\{\partial_1|_p, \dots, \partial_n|_p\}$  such that  $\partial_j|_p$  maps the differentiable function  $f$  at  $p$  into the number  $(\partial f/\partial x^j)|_{\lambda(0)}$ . The tangent vectors at  $p$  form an  $n$ -dimensional vector space  $T_p(\mathcal{M})$  over  $R^1$ . This is the *tangent manifold* at  $p$ .

If  $\{\mathbf{e}_a|_p\}$  ( $a = 1$  to  $n$ ) are any set of  $n$  vectors at  $p$  which are linearly independent, then any vector  $\mathbf{V}|_p \in T_p$  can be written as  $\mathbf{V}|_p = V^a|_p \mathbf{e}_a|_p$ , where the numbers  $V^a|_p$  are the *components* of  $\mathbf{V}|_p$  with respect to the basis  $\{\mathbf{e}_a|_p\}$  of 1-vectors at  $p$ .

Finally, the metric tensor  $\mathcal{G}$  at  $p \in \mathcal{M}$  is a bilinear functional from  $T_p \times T_p$  to  $R$ , i.e., a tensor of type  $(0, 2)$ . We shall denote the inner product of 1-

vectors  $\mathbf{u}, \mathbf{v}$  at  $p$  by  $(\mathbf{u} \cdot \mathbf{v})|_p$ . In particular, if  $\{\mathbf{e}_1|_p, \dots, \mathbf{e}_n|_p\}$  is any basis for 1-vectors at  $p$ , we shall write

$$g_{ij}|_p := (\mathbf{e}_i \cdot \mathbf{e}_j)|_p, \quad (7.1)$$

so

$$(\mathbf{u} \cdot \mathbf{v})|_p = (g_{ij}u^i v^j)|_p.$$

## Cotangent Manifold at a Point

Having introduced the vector space  $T_p(\mathcal{M})$ , we can now proceed to introduce the so-called *cotangent manifold*  $T_p^*(\mathcal{M})$  of differential 1-forms at  $p$ . A 1-form  $\omega|_p$  at  $p$  is a real-valued linear functional on the space  $T_p(\mathcal{M})$  of tangent vectors at  $p$ . If  $\mathbf{X}|_p$  is a vector at  $p$ , the number into which  $\omega|_p$  maps  $\mathbf{X}|_p$  will be written as  $(\mathbf{X}\omega)|_p$ , with the differential form written *to the right* of the vector upon which it operates.

Given any basis  $\{\mathbf{e}_i|_p\}$  of 1-vectors at  $p$ , one can define in a unique fashion a corresponding basis  $\{e^i|_p\}$  for 1-forms at  $p$ . The linear functional  $e^j|_p$  maps any vector  $\mathbf{X}|_p$  at  $p$  to the number  $X^j|_p$ , i.e., the  $j$ th component of  $\mathbf{X}$  with respect to the basis  $\{e_i\}$ . Thus

$$(\mathbf{e}_i e^j)|_p = \delta_i^j,$$

where  $\delta_i^j$  is the Kronecker delta function, equal to 1 for  $i = j$  and equal to 0 otherwise. The set of all 1-forms at  $p$  constitutes another  $n$ -dimensional vector space at  $p$ , the dual space  $T_p^*(\mathcal{M})$  of the tangent space  $T_p(\mathcal{M})$ . This is the so-called *cotangent manifold* at  $p$ .

If the coordinates of a particular chart in the neighborhood of  $p$  are denoted by  $x^1, \dots, x^n$ , then the differentials  $dx^1|_p, \dots, dx^n|_p$  at  $p$  form a basis for one-forms which is dual to the basis  $\boldsymbol{\partial}_1|_p, \dots, \boldsymbol{\partial}_n|_p$  for tangent vectors at  $p$ .

If the tangent manifold is endowed with an inner product such that  $g_{ij}|_p$  is given by Eq. (7.1), then the notion of an inner product can be extended to the cotangent manifold, by defining

$$(dx^i \cdot dx^j)|_p = g^{ij}|_p,$$

where  $g^{ij}|_p$  is the inverse metric tensor, i.e.,

$$(g_{ij}g^{jk})|_p = \delta_i^k.$$

Strictly speaking a different “dot” should be used here. Later, to avoid confusion, we shall introduce different symbols for the (Grassmann) inner products of tangent vectors and differential forms. One may use the metric tensor and its inverse to lower and raise indices freely, including those on the basic tangent vectors and those on the basic 1-forms.

## Exterior Products at Point $p$

At this point Hawking and Ellis consider tensor products of tangent vectors and tensor products of one-forms. Except for the metric tensor  $\mathcal{G}$ , we shall find it sufficient to restrict attention to completely skew symmetrical tensor products, i.e., to exterior products of tangent vectors and exterior products of 1-forms. Thus, we shall consider bivectors such as  $(\mathbf{e}_1 \wedge \mathbf{e}_2)|_p$ , and more generally  $r$ -vectors, and we shall consider 2-forms such as  $(e^1 \wedge e^2)|_p$ , and more generally  $s$ -forms. Of course, the  $\wedge$  symbol involved in the two products is really different, just as the  $+$  sign used to denote vector addition is different from the one used to denote ordinary addition. Because in calculations I often use exterior products of differential forms, I have found it convenient to suppress completely the  $\wedge$  symbol whenever exterior products of differential forms are involved. I have also found it convenient to introduce more general objects such as, for example

$$(e^1 e^2 \mathbf{e}_1 \wedge \mathbf{e}_3)|_p,$$

that is, objects which are tensor products of  $s$ -forms and  $r$ -vectors. We shall refer to such an object as an  $(s, r)$ -tensor. Such objects are members of the manifold  $\Lambda_s^r(p)$ . In particular,  $\Lambda_0^1(p) = T_p$  and  $\Lambda_1^0(p) = T_p^*$ .

So far we have only introduced tangent vectors and differential forms at a point. However, we can easily introduce the notion of *vector fields* or more generally  $\Lambda_s^r$ -fields on a set  $\mathcal{U} \subseteq \mathcal{M}$ .

## The Differential Operator $d$

The exterior differential operator  $d$  is an object that has the attributes of a one-form as well as the attributes of a differential operator, viz., linearity and the Leibnitz property. Thus, it resembles the vector differential operator  $\vec{\nabla}$  encountered in undergraduate courses in mechanics and electromagnetism, which simultaneously possesses the attributes of a vector and those of a differential operator.

We shall define the action of  $d$  upon any arbitrary member of  $\Lambda_s^r$ .<sup>2</sup> This can be done most easily by acknowledging that  $d$  is a linear operator with the Leibnitz property, and then specifying the action of  $d$  upon basic objects.

- (1) Upon any scalar function  $f(\mathbf{x})$  the operator  $d$  will have the effect

$$df = dx^a \frac{\partial f}{\partial x^a} ,$$

i.e., it yields a 1-form whose components are the various partial derivatives of the function  $f$  with respect to the coordinates.

- (2) Upon any of the 1-forms  $dx^a$  of a coordinate basis, it yields zero, i.e.,  $d(dx^a) = 0$ .
- (3) Upon any of the one-vectors  $\partial_b$  of a coordinate basis, it yields a member of  $\Lambda_1^1$ , which we shall express in the form

$$d\partial_b = \Gamma_{ab}{}^c dx^a \partial_c ,$$

where the coefficients  $\Gamma_{ab}{}^c$  are called the components of the (affine) *connection* or the *Christoffel symbols of the second kind* with respect to the chosen chart.

There are a number of different theories of gravitation that differ from one another by having different connections. The Einstein theory has a *symmetric connection*, i.e.,

$$\Gamma_{ab}{}^c = \Gamma_{ba}{}^c .$$

On the other hand, the less fashionable Einstein-Cartan theory has an asymmetrical connection, the asymmetry being associated with a torsion field not present in the standard Einstein theory. It should be noted that symmetry of the connection can be expressed in the alternative manner

$$d\mathbb{I} = -dx^a d\partial_a = 0 ,$$

where

$$\mathbb{I} := dx^a \partial_a$$

is the unit (1, 1)-tensor.

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<sup>2</sup>Here we diverge from the treatment of Hawking and Ellis, who distinguish between the  $d$  operator of exterior differentiation of forms, and the  $\nabla$  operator of covariant differentiation of tensor fields.

## Covariant Derivative

For an arbitrary 1-vector field  $\mathbf{X}$ , the  $(1, 1)$ -tensor  $d\mathbf{X}$  is called the *covariant derivative* of  $\mathbf{X}$ . I think of this as an abstract geometrical object, which possesses components relative to any specified basis for vectors and forms. If  $\mathbf{X} = X^b \boldsymbol{\partial}_b$ , then

$$\begin{aligned} d\mathbf{X} &= dX^b \boldsymbol{\partial}_b + X^b d\boldsymbol{\partial}_b \\ &= dX^c \boldsymbol{\partial}_c + X^b \Gamma_{ab}{}^c dx^a \boldsymbol{\partial}_c \\ &= \left( \frac{\partial X^c}{\partial x^a} + X^b \Gamma_{ab}{}^c \right) dx^a \boldsymbol{\partial}_c . \end{aligned}$$

Thus,

$$d\mathbf{X} = (\nabla_a X^c) dx^a \boldsymbol{\partial}_c ,$$

where the components of the covariant derivative are given by

$$\nabla_a X^c := \frac{\partial X^c}{\partial x^a} + X^b \Gamma_{ab}{}^c .$$

I should like to emphasize again that one should think of  $d\mathbf{X}$  as an abstract geometrical object, just as one should think of  $\mathbf{X}$  as an abstract geometrical object. The objects differ, of course, for one is a member of  $\Lambda_1^1$  and the other is a member of  $\Lambda_0^1$ , but either object can be expressed in terms of specific coordinates whenever that is desired.

## Evaluation of the Christoffel Symbols

We shall close this chapter by evaluating the Christoffel symbols  $\Gamma_{ab}{}^c$  in terms of partial derivatives of the metric tensor  $g_{ab}$ . Recall that

$$d\boldsymbol{\partial}_b = \Gamma_b{}^c \boldsymbol{\partial}_c ,$$

where

$$\Gamma_b{}^c := dx^a \Gamma_{ab}{}^c ,$$

and where (in the Einstein theory) the Christoffel symbol  $\Gamma_{ab}{}^c$  is symmetric in the lower two indices.

Consider the differential of the metric tensor

$$g_{ab} := \boldsymbol{\partial}_a \cdot \boldsymbol{\partial}_b .$$

We have

$$\begin{aligned}
 dg_{ab} &= d(\boldsymbol{\partial}_a \cdot \boldsymbol{\partial}_b) \\
 &= d\boldsymbol{\partial}_a \cdot \boldsymbol{\partial}_b + d\boldsymbol{\partial}_b \cdot \boldsymbol{\partial}_a \\
 &= \Gamma_a^c \boldsymbol{\partial}_c \cdot \boldsymbol{\partial}_b + \Gamma_b^c \boldsymbol{\partial}_c \cdot \boldsymbol{\partial}_a \\
 &= \Gamma_a^c g_{cb} + \Gamma_b^c g_{ca} \\
 &= \Gamma_{ab} + \Gamma_{ba} .
 \end{aligned}$$

The symmetric part of the matrix  $\Gamma_{ab}$  is completely determined by the metric tensor  $g_{ab}$ . If we can determine the skew-symmetric part of  $\Gamma_{ab}$  we shall be finished.

Let the *Ricci rotation matrix* be defined as the skew symmetrical part of  $\Gamma$ , i.e., as

$$\Omega_{ab} := \frac{1}{2}(\Gamma_{ab} - \Gamma_{ba}) .$$

Now,  $dx^a \Gamma_{ab} = 0$ , because of the assumed symmetry of the Christoffel symbol  $\Gamma_{ab}^c$  in the lower two indices. (Note that  $\Gamma_{ab}$  is a one-form, so an exterior product is implied in the above relation.) Substituting  $\Gamma_{ab} = \frac{1}{2}dg_{ab} + \Omega_{ab}$ , we obtain the important result

$$dx^a \Omega_{ab} = -\frac{1}{2}dx^a dg_{ab} .$$

The two-form on the right side can easily be evaluated. It then remains to find a skew-symmetric matrix  $\Omega$  of one-forms such that the left side equals the given right side.

**Ex. 13** Show that the solution of the above equation is uniquely given by

$$\Omega_{ab} = \frac{1}{2}dx^c \left( \frac{\partial g_{cb}}{\partial x^a} - \frac{\partial g_{ca}}{\partial x^b} \right) .$$

From this show that

$$\Gamma_{ab}^c = \frac{1}{2}g^{cd} \left( \frac{\partial g_{ad}}{\partial x^b} + \frac{\partial g_{bd}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^d} \right) .$$