A RELATIVISTIC EXPLANATION OF GRAVITATIONAL ACCELERATION OF FALLING BODIES NEAR THE EARTH

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It is well known that gravitation (which will be used to refer to the effect of falling toward the earth) can be explained in two manners:

1. A "force of gravity" (which will be used to signify a force of attraction between masses) causes the bodies to approach each other.

2. Space surounding great masses is non-Euclidean. A body in this space follows that path which is easiest to follow (the geodesic). In the following article I show that the body will move toward the great mass which is contorting the space.

Both these methods are quite valid and yield the same results, but I prefer the latter since it is the easier to propound mathematically. My project consists of applying the curved space method to the study of bodies falling from rest, a very common occurrence. The exploitation of this project necessitated the study of the most modern, the most powerful mathematics ever conceived, the tensor calculus. Therefore, in order to abridge my work as much as possible I refer the reader to "The Einstein Theory of Relativity" by L. R. Lieber for the background necessary to understand the following exposition.

I. Gathering the Material with which to Find the Gravitational Acceleration of a Falling Body

The values of the Christoffel symbols of our four-dimensional non-Euclidean space-time are as follows: (See Appendix A)

$$\{11,1\} = \frac{1}{2}\lambda' \qquad \{14,4\} = \tfrac{1}{2}\nu' \qquad \{33,1\} = -r\sin^2\theta e^{-\lambda}$$
$$\{12,2\} = \frac{1}{r} \qquad \{22,1\} = -re^{-\lambda} \qquad \{33,2\} = -\sin\theta\cos\theta$$
$$\{13,3\} = \frac{1}{r} \qquad \{23,3\} = \cot\theta \qquad \{44,1\} = \frac{1}{2}e^{\nu-\lambda}\nu'$$

The equation of the geodesic of this space is defined by

$$\frac{d^2x_\sigma}{ds^2} + \{\alpha\beta,\sigma\}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 0.$$

Substituting the appropriate values for $\sigma = 1$ to 4

(1) $\frac{d^2r}{ds^2} + \frac{1}{2}\lambda'\left(\frac{dr}{ds}\right)^2 - re^{-\lambda}\left(\frac{d\theta}{ds}\right)^2 - r\sin^2\theta e^{-\lambda}\left(\frac{d\phi}{ds}\right)^2 + \frac{1}{2}e^{\nu-\lambda}\nu'\left(\frac{dt}{ds}\right)^2 = 0$

(2) $$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0$$

(3) $$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0$$

(4) $$\frac{d^2t}{ds^2} + \nu'\frac{dr}{ds}\frac{dt}{ds} = 0$$

These equations represent the path of a body in space under the influence of no forces. r, θ, ϕ and t are polar coodinates.

II. First Method for Finding the Gravitational Acceleration of a Freely Falling Body

I reasoned that since $\theta = 0$ is a solution of equation (2), then the motion of the body would be along a line joining the center of the body to the center of the earth. Furthermore, I solved equations (1) and (4) for dr/ds and dt/ds in order to find out the nature of the motion along that straight line. I found that

$$\frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} = 1 - \frac{2m}{r} \qquad \text{(See Appendix B)}$$

Differentiating

$$\frac{d^2r}{dt^2} = \frac{2m}{r^2}\frac{dr}{dt}$$

For the earth dr/dt is very nearly equal to one since the density m/r is very small. The acceleration of a falling body is therefore given by

$$\frac{d^2r}{dt^2} = \frac{2m}{r^2} \qquad \text{Answer.}$$

However, this method did not withstand further investigation. (See Appendix C) Obviously there was a flaw in the reasoning.

III. Second Method for Finding the Gravitational Acceleration

After several weeks of toiling over the problem, I discovered what I believed to be the cause of the first method's not working. Investigating equation (3) I found that since $\cot\theta =$ infinity and $d\theta/ds = 0$, the equation is indeterminate.

Then I recalled that Einstein had found a different set of solutions for the four simultaneous equations of the geodesic. I wondered if I could find the acceleration just as well be using his values.

Einstein let the solution of equation (2) by $\theta = \frac{\pi}{2}$. Substituting in the remaining equations, it is easy to find by integration that

$$\frac{d\phi}{ds} = \frac{h}{r^2} \text{ and } \frac{dt}{ds} = \frac{c}{\gamma}, \text{ (See Appendix D)}$$

but I do not believe he computed dr/ds. I proceeded to find dr/ds and the gravitational acceleration as follows:

Einstein gives the square of the line element in this form.

$$\begin{aligned} ds^2 &= -\frac{1}{\gamma}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 + \gamma dt^2 \\ 1 &= \gamma\left(\frac{dt}{ds}\right)^2 - \frac{1}{\gamma}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\theta}{ds}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 \\ 1 &= \frac{c^2}{\gamma} - \frac{1}{\gamma}\left(\frac{dr}{ds}\right)^2 - \frac{h^2}{r^2} \\ \left(\frac{dr}{ds}\right)^2 &= c^2 - \gamma - \frac{h^2}{r^2}\gamma = c^2 - \gamma \end{aligned}$$

A body falling from rest travels along a line connecting the center of the body with the center of the earth. Therefore, ϕ is a constant and h equals zero, for $d\phi/ds = h/r^2$.

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{dr}{ds}\right)^2\left(\frac{ds}{dt}\right)^2 = \left(c^2 - \gamma\right)\frac{\gamma^2}{c^2} = \gamma^2 - \frac{\gamma^2}{c^2}$$

$$2\left(\frac{dr}{dt}\right)\frac{d^2r}{dt^2} = 2\gamma\frac{d\gamma}{dr}\frac{dr}{dt} - \frac{3\gamma^2}{c^2}\frac{d\gamma}{dr}\frac{dr}{dt}$$

$$\frac{d^2r}{dt^2} = \gamma\frac{d\gamma}{dr} - \frac{3\gamma^2}{2c^2}\frac{d\gamma}{dr} = \left(1 - \frac{3}{2c^2}\right)\frac{d\gamma}{dr}$$

γ is very nearly equal to one. Furthermore, c represents the velocity of light, which equals one kilometer of space per kilometer of time. (See Appendex E) In addition, since $\gamma = 1 - \frac{2m}{r}$, the derivative is equal to $\frac{2m}{r^2}$.

$$\frac{d^2r}{dt^2} = -\frac{m}{r^2} \text{ Answer.}$$

IV. Conclusion

Thus, by studying falling bodies from the viewpoint of curved space instead of by an assumed "force of attraction," two things have been learned:

1. A body falling from rest will move in a straight line joining the center of the body with the center of the earth.

2. The body will approach the earth with an acceleration given by

$$a = -\frac{m}{r^2}$$

These results were obtained also by Newton, but he attributed the approaching of the smaller to the larger body to a force in some way characteristic of matter.

It should be made clear that both methods, the geometric and the dynamic, are equally valid. The only essential difference in Newton's and Einstein's theories is that Newton assumes time invariant while Einstein claims time is <u>not</u> invariant. Of the two methods, however, I prefer Einstein's geometric method because it lends itself so easily to mathematical analysis. For instance, if a body is not falling from rest, i.e., has an initial velocity, I found it easy to show that the gravitational acceleration would <u>not</u> be the same as that of a body released from rest. The acceleration in general would be

$$a = -\frac{m}{r^2} - 3m\left(\frac{d\phi}{ds}\right)^2 + r\left(\frac{d\phi}{ds}\right)^2.$$

In this equation $d\phi/ds$ represents the instantaneous speed of the radius vector.

Certainly the dynamic "force of gravity" method cannot yield the corresponding result as facile as this geometric method. There is the essential point I have tried to express - namely, curved space explains the facts more easily than does the dynamic theory of Newton. To further illustrate this I shall soon try to explain gravitation goemetrically while assuming time invariant. Using this method I should obtain exactly the same formulas as Newton did. Conversely, it should be possible, but extremely difficult, to explain gravitation dynamically while assuming time not invariant. The result should be exactly like Einstein's. These will be great tasks, but nevertheless they are tasks which I desire to accomplish.

Bibliography

I acquired the mathematics necessary for this project largely by studying from (1) "Elements of the Differential and Integral Calculus" by Granville, Smith and Longley, and (2) "Matrix and Tensor Calculus" by Michal.

I owe my interest in the mathematical theory of relativity to a fine book by Lillian R. Lieber, (3) "The Einstein Theory of Relativity."

Appendix A:

The Christoffel Symbols of the Space-Time Continuum

The Christoffel symbols were determined by Einstein from the expression for the square of the line element in the following manner:

$$ds^2 = -e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + e^\nu dt^2$$
$$g_{11} = -e^\lambda, g_{22} = -r^2, g_{33} = -r^2 \sin^2\theta, g_{44} = e^\nu.$$
$$\{12, 2\} = \frac{1}{2}\frac{\partial}{\partial r}\log g_{22} = \frac{1}{2}\frac{\partial}{\partial r}\log(-r^2).$$
$$\therefore \{12, 2\} = \frac{-2r}{2(-r^2)} = \frac{1}{r}.$$

Many of the Christoffel symbols were equal to zero, but 13 were not. Of these 13 only 9 were independent because a Christoffel symbol is symmetrical. For instance $\{21, 2\} = \{12, 2\}$. These symbols characterize the space which we are studying.

Appendix B

How dr/dt Was Found

Equations (1) and (2) are satisfied by $dr/ds = 1$ and $dt/ds = 1/\gamma$. And $1/\gamma = e^\lambda$ according to Einstein. Substituting in (1)

$$\frac{1}{2}\lambda' + \frac{1}{2}e^{\nu-\lambda}\nu' e^{2\lambda} = 0$$

Einstein states that $\nu = -\lambda$ and therefore $\nu' = -\lambda'$ also.

$$\frac{1}{2}\lambda' + \frac{1}{2}\nu' = 0$$

$$0 = 0 \text{ This equation is satisfied.}$$

Making the necessary substitutions in (4)

$$e^{\lambda}\lambda' + \nu' e^{\lambda} = 0$$

$$0 = 0 \text{ This equation is satisfied.}$$

So $dr/dt = \frac{dr}{ds}\frac{ds}{dt} = \gamma = 1 - \frac{2m}{r}$.

Appendix C

Difficulties in the First Method

The values $\theta = 0$, $dr/ds = 1$, and $dt/ds = 1/\gamma$ do not check when substituted in the expression for the square of the line element.

$$\begin{aligned}
ds^2 &= -\frac{1}{\gamma}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + \gamma dt^2 \\
1 &= -\frac{1}{\gamma}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\theta}{ds}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 + \gamma\left(\frac{dt}{ds}\right)^2 \\
1 &= -\frac{1}{\gamma} + \frac{1}{\gamma} \\
1 &= 0 \text{ which is impossible.}
\end{aligned}$$

Furthermore, since the acceleration is positive, it must be directed away from the earth. That does not agree with experimental knowledge.

Appendix D

Determining $d\phi/ds$ and dt/ds

Since $\theta = \frac{\pi}{2}$, $\cot\theta = 0$ and equation (3) becomes

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} = 0$$

Let $d\phi/ds = \phi'$

$$\frac{1}{\phi'}\frac{d\phi'}{ds} + \frac{2}{r}\frac{dr}{ds} = 0$$

$$\log\phi' + \log r^2 = \log h$$

$$\therefore \frac{d\phi}{ds} = \frac{h}{r^2}$$

letting $dt/ds = t'$, equation (1) becomes

$$\frac{1}{t'}\frac{dt'}{ds} + \nu'\frac{dr}{ds} = 0$$
$$\log t' + \log e^{\nu} = \log c$$
$$\therefore \frac{dt}{ds} = \frac{c}{e^{\nu}} = \frac{c}{\gamma}$$

Appendix E

Why the Constant "c" is the Velocity of Light

In the special theory of relativity ds^2 has the form

$$ds^2 = c^2dt^2 - \left[dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right]$$

where $c =$ the velocity of light. Now, the same form is necessarily carried over into the non-Euclidean expression

$$ds^2 = \gamma c^2dt^2 - \left[\frac{1}{\gamma}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right]$$

In both expressions c is equal to one since the speed of light is one kilometer of space per kilometer of time (1/300,000 sec). Integration of

$$\frac{d^2t}{ds^2} + \nu'\frac{dr}{ds}\frac{dt}{ds} = 0$$

seems to reintroduce that same constant. The dimensionality of c^2dt^2 is $\left(\frac{L^2}{T^2}T^2\right) = L^2$. This is logical because to be added to the other L^2's, i.e.. γdr^2, $r^2d\theta^2$, and $r^2\sin^2\theta d\phi^2$, c^2dt^2 must be of dimensionality L^2. Also, in the equation

$$ds^2 = \gamma c^2dt^2 - \left[\frac{1}{\gamma}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right]$$

dt^2 must be of dimensionality L^2 or c^2T^2 where $c =$ the velocity of light. But if we use Einstein's system of dimensionality, the time is expressed in kilometers; so, dt^2 is in kil^2. That agrees with $dt/ds = c/\gamma$, for $dt^2 = \frac{c^2}{\gamma^2}ds^2$ which is of dimensionality L^2 or kil^2. The constant of integration "c" is the velocity of light.