Determining a stationary axisymmetric electrovac spacetime from its axis data

Frederick J. Ernst
FJE Enterprises, 511 CR 59, Potsdam, NY 13676

Abstract

We review the history of the problem of determining a stationary axisymmetric electrovac spacetime from its axis data $\mathcal{E}(z,0)$ and $\Phi(z,0)$.

1 Introduction

The idea of using the axis values $\mathcal{E}(z,0)$ and $\Phi(z,0)$ of the complex Ernst potentials $\mathcal{E}(z,\rho)$ and $\Phi(z,\rho)$ in order to identify Kinnersley-Chitre (KC) transformations [1] that might be used to generate one stationary axisymmetric electrovac spacetime from another one was promulgated by Hauser and Ernst in 1980. Indeed, in the talk that I presented [2] at the GR9 Meeting in Jena, I displayed our "axis relations" and used them to identify a KC transformation that could be used to generate one electrovac solution from another, while preserving the value of $f := \text{Re }\mathcal{E} + \Phi^*\Phi$. I also remarked that, in the case of this particular transformation, solving the Hauser-Ernst homogeneous Hilbert problem (HHP) [3] reduced to solving a nonmatrix linear integral equation.

Figuring out what KC transformation might be used to generate a vacuum metric with specified values of $\mathcal{E}(z,0)$ was even easier. Regretfully, my colleague, Isidore Hauser, did not mention in his presentation at Jena the specific KC transformation that we advocated employing in practical calculations, although it did appear in our subsequent paper [4] concerned with the proof of the Geroch conjecture, where we demonstrated [following Eq.

(69)] that the transformation

$$u(t) = \begin{pmatrix} f(\tau,0)^{1/2} & t\chi(\tau,0)f(\tau,0)^{-1/2} \\ 0 & f(\tau,0)^{-1/2} \end{pmatrix}, \tag{1}$$

with $\tau := 1/(2t)$, can be used to generate a vacuum metric with any specified axis data $\mathcal{E}(\tau,0) = f(\tau,0) + i\chi(\tau,0)$. In fact, this transformation played a key role in proving the Geroch conjecture.

2 A linear system for the Ernst equations

Our investigations involved a linear system

$$dF(t) = \Gamma(t)\Omega F(t), \quad \Gamma(t) := t\lambda(t)^{-2}[(1-2tz) + 2t\rho\star)]dF^{(1)} \tag{2}$$

for the Ernst equations [5]

$$\begin{aligned} (\mathrm{Re}\ \mathcal{E} + \Phi^*\Phi)\nabla^2\mathcal{E} &= (\nabla\mathcal{E} + 2\Phi^*\nabla\Phi)\cdot\nabla\mathcal{E}, & (3)\\ (\mathrm{Re}\ \mathcal{E} + \Phi^*\Phi)\nabla^2\Phi &= (\nabla\mathcal{E} + 2\Phi^*\nabla\Phi)\cdot\nabla\Phi, & (4)\end{aligned}$$

that we inferred from earlier work [1] of Kinnersley and Chitre. In the above equation

$$\lambda(t) := \sqrt{(1-2tz)^2 + (2t\rho)^2}, \quad \lambda(0) := +1, \tag{5}$$

and

$$F^{(1)} := \begin{pmatrix} H & \phi \\ 2iL & 2iK \end{pmatrix}, \tag{6}$$

where H is the 2×2 matrix generalization of Kinnersley and Chitre of the Ernst potential $\mathcal{E}$, ϕ is their 2×1 matrix generalization of the Ernst potential Φ, and L and K are 1×2 and 1×1 matrix potentials that satisfy the equations

$$dL = \phi^\dagger\epsilon dH, \quad dK = \phi^\dagger\epsilon d\phi, \tag{7}$$

respectively. We also employed the constant matrices

$$\epsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega := \begin{pmatrix} i\epsilon & 0 \\ 0 & 1 \end{pmatrix}, \tag{8}$$

and it was to be understood that $F(t)$ was to be selected so that

$$\begin{aligned} F(0) &= \Omega, && (9)\\ \dot{F}(0) &= F^{(1)}, && (10)\\ \det F(t) &= -\lambda(t)^{-1}, && (11) \end{aligned}$$

and

$$F(t)^{\dagger}\left[\Omega - t\Omega(H + H^{\dagger}\right] F(t) = \Omega, \tag{12}$$

where dots denote derivatives with respect to the spectral parameter t.

3 The axis relations

In the Hauser-Ernst gauge, the F-potential $F(t)$ has in the complex t-plane only the singularities of $\lambda(t)$, i.e., branch points of index 1/2 at $t = 1/[2(z \pm i\rho)]$. When $\rho \to 0$, these branch points collapse to a simple pole at $t = 1/(2z)$. Because of this it turned out to be possible to integrate the linear system (2) along the symmetry axis.

Noting that

$$\star dz = -d\rho, \quad \star d\rho = dz, \tag{13}$$

we may write Eq. (2) in the more explicit form

$$\begin{aligned} F_z(t) &= t\lambda(t)^{-2}\left[(1-2tz)F_z^{(1)} + 2t\rho F_\rho^{(1)}\right]\Omega F(t), && (14)\\ F_\rho(t) &= t\lambda(t)^{-2}\left[(1-2tz)F_\rho^{(1)} - 2t\rho F_z^{(1)}\right]\Omega F(t). && (15) \end{aligned}$$

On the axis, these equations reduce to

$$\begin{aligned} F_z(t) &= t(1-2tz)^{-1}F_z^{(1)}\Omega F(t), && (16)\\ F_\rho(t) &= t(1-2tz)^{-1}F_\rho^{(1)}\Omega F(t), && (17) \end{aligned}$$

where, however, it is known that

$$F^{(1)}(z,0) = \begin{pmatrix} 0 & 0 & 0 \\ -2iz & \mathcal{E}(z,0) & \Phi(z,0) \\ 0 & 0 & 0 \end{pmatrix}, \quad F_\rho^{(1)}(z,0) = 0. \tag{18}$$

Therefore, on the axis, $F(t)$ has constant first and third rows. Using Eqs. (16) and (17) together with conditions (9) to (12), it was found that, on the axis,

$$F(t) = \begin{pmatrix} 0 & i & 0 \\ -\frac{i}{1-2tz} & \frac{t\mathcal{E}(z,0)}{1-2tz} & \frac{t\Phi(z,0)}{1-2tz} \\ 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

Of course, $F_0(t)$ has a similar form in terms of $\mathcal{E}_0(z,0)$ and $\Phi_0(z,0)$. Thus,

$$F_0(t)^{-1} = \begin{pmatrix} -t\mathcal{E}_0(z,0) & i(1-2tz) & -it\Phi_0(z,0) \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

However, according to our HHP,

$$X_-(t) := F(t)u(t)F_0(t)^{-1} \tag{21}$$

cannot have a pole at $t = 1/(2z)$. To avoid such a pole in the 21 and 23 elements of $X_-(t)$, evaluated on the axis, two conditions must be satisfied. In [2] these conditions were expressed in the form

$$\begin{pmatrix} -i & t\mathcal{E}(\tau,0) & t\Phi(\tau,0) \end{pmatrix} u(t) \begin{pmatrix} -t\mathcal{E}_0(\tau,0) & -it\Phi_0(\tau,0) \\ -i & 0 \\ 0 & 1 \end{pmatrix} = 0, \tag{22}$$

where $\tau := 1/(2t)$.

By solving the axis relations for $u(t)$, subject to the conditions

$$u(t)^\dagger \Omega(t) u(t) = \Omega(t), \quad \det u(t) = +1, \tag{23}$$

where

$$\Omega(t) := \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & t/2 \end{pmatrix}, \tag{24}$$

one can determine a family of KC transformation matrices $u(t)$, using which one can transform the initial metric with axis data $\mathcal{E}_0(z,0)$ and $\Phi_0(z,0)$ into a final metric with axis data $\mathcal{E}(z,0)$ and $\Phi(z,0)$. While, in practice, Minkowski space is often employed for the initial spacetime, with $\mathcal{E}_0 = 1$ and $\Phi_0 = 0$, any initial metric can be employed.

The transformations that we advocated using to generate any stationary axisymmetric electrovac solution from Minkowski space were the transformation [4]

$$u_{vac}(t) = \begin{pmatrix} f(\tau,0)^{1/2} & t\chi(\tau,0)f(\tau,0)^{-1/2} & 0 \\ 0 & f(\tau,0)^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{25}$$

followed by the transformation [2]

$$u_{em}(t) = \begin{pmatrix} 1 & -i\Phi(\tau,0)^*\Phi(\tau,0) & -it\Phi(\tau,0) \\ 0 & 1 & 0 \\ 0 & 2\Phi(\tau,0)^* & 1 \end{pmatrix}. \tag{26}$$

The first transformation produces from Minkowski space a vacuum metric with axis data $\mathcal{E}(z,0) = f(z,0) + i\chi(z,0)$, while the second transformation preserves $f(z,0)$ and yields an electrovac metric with axis data $\mathcal{E}(z,0) = f(z,0) - \Phi(z,0)^*\Phi(z,0)$ and $\Phi(z,0)$.

It is, of course, possible to factorize the first transformation into two factors,

$$u_{vac}(t) = u_{HKX}(t)u_W(t), \tag{27}$$

where

$$u_{HKX}(t) = \begin{pmatrix} 1 & t\chi(\tau,0) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad u_W(t) = \begin{pmatrix} f(\tau,0)^{1/2} & 0 & 0 \\ 0 & f(\tau,0)^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{28}$$

The first transformation produces a Weyl metric with axis data $f(z,0)$, while the second transformation, acting upon that Weyl metric, yields a vacuum metric with axis data $\mathcal{E}(z,0) = f(z,0) + i\chi(z,0)$. Solving the HHP corresponding to the latter "triangular" transformation involves solving a Fredholm integral equation of the second kind.

Using the Kinnersley-Chitre infinite hierarchy approach, Hoenselaers, Kinnersley and Xanthopoulos [6] already used such transformations in practical calculations in 1979. In particular, they showed that the F-potential of the Weyl metric that is produced when the first transformation, with $f(z,0) = e^{2\psi(z,0)}$, acts upon Minkowski space is given by

$$F(t) = e^{-\psi}F^{MS}(t)e^{-\Psi(t)}, \tag{29}$$

where $\Psi(t)$ is determined from the Weyl potential ψ by solving the first order linear differential equation

$$d\Psi(t) = \frac{1 - 2tz + 2t\rho\star}{\lambda(t)}d\psi. \tag{30}$$

They also employed the second transformation to construct vacuum solutions with axis data $\chi(z,0)$ that were rational functions of z. In this case, all the matrix elements of $u(t)$ were meromorphic functions, and they could, therefore, decompose the triangular transformation into simpler triangular factors, each of which involved a pole at a single location in the t-plane.

If one evaluates the product of the transformations (25) and (26), one obtains the transformation

$$u_S(t) = \begin{pmatrix} f(\tau,0)^{\frac{1}{2}} & [t\chi(\tau,0) - i\Phi(\tau,0)^*\Phi(\tau,0)]f(\tau,0)^{-\frac{1}{2}} & -it\Phi(\tau,0) \\ 0 & f(\tau,0)^{-\frac{1}{2}} & 0 \\ 0 & 2\Phi(\tau,0)^* f(\tau,0)^{-\frac{1}{2}} & 1 \end{pmatrix}, \tag{31}$$

upon which Sibgatullin [7] based his method of generating electrovac spacetimes from axis data.

4 Sibgatullin's integral equation

In describing the reasoning that led Sibgatullin to his integral equation, I shall employ the τ-plane representation, as it is a little simpler than the t-plane representation that Sibgatullin employed. At the Retzbach meeting in 1983, we announced a change in our conventions concerning $F^{(1)}$, which we began to write in the form

$$F^{(1)} := \begin{pmatrix} H & 2\phi \\ iL & 2iK \end{pmatrix}, \tag{32}$$

instead of the form given in Eq. (6). This change, which took place simultaneously with and was motivated by the adoption of a new τ-plane representation of our HHP, also caused changes involving factors of two in the form taken by the axis relations, concerning which I hereby notify readers of our papers that appeared from 1983 onward.

Note that in our τ-plane representation the P-potential is defined by

$$P(\tau) := F(t)\Delta(t), \quad \Delta(t) := \begin{pmatrix} \sqrt{2}t & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{33}$$

where $F(0) = 0$ and $\dot{F}(0) = F^{(1)}$, with $F^{(1)}$ given by Eq. (32), and the transformation (31) assumes the form

$$v_S(\tau) =$$

$$\begin{pmatrix} f(\tau,0)^{\frac{1}{2}} & [\chi(\tau,0) - i\Phi(\tau,0)^*\Phi(\tau,0)]f(\tau,0)^{-\frac{1}{2}} & -\sqrt{2}i\Phi(\tau,0) \\ 0 & f(\tau,0)^{-\frac{1}{2}} & 0 \\ 0 & \sqrt{2}\Phi(\tau,0)^* f(\tau,0)^{-\frac{1}{2}} & 1 \end{pmatrix} \quad (34)$$

In our research Hauser and I were generally motivated by the desire to prove general theorems rather than to construct specific solutions. Of course, in establishing equivalence of our HHP to various integral equations, and especially to a Fredholm equation of the second kind, we provided an approach that could be used to construct solutions with arbitrarily specified axis data. While our objectives required that we keep the Geroch group and the Kinnersley-Chitre transformations in the foreground, if one is only interested in constructing specific solutions, one might as well start with Minkowski space and employ a transformation such as (34). Indeed, one can even cast aside any concern with elements of the 3 × 3 Hauser-Ernst potential $P(\tau)$ that are just needed if one wishes to apply additional KC transformations to the constructed spacetime, concentrating instead upon those elements that are directly involved in the determination of the complex Ernst-potentials $\mathcal{E}$ and Φ.

In the Hauser-Ernst gauge the P-potential is holomorphic everywhere in the τ-plane except at the two branch points $\tau = z \pm i\rho$ and on the cut connecting these two branch points. Therefore, if we restrict attention to spacetime points such that the branch cut $\mathcal{L}$ lies entirely outside the positively oriented closed contour Λ, we have

$$P(\tau) = \frac{1}{2\pi i}\int_\Lambda d\sigma \frac{P(\sigma)}{\sigma - \tau} \quad (35)$$

for any τ inside the closed contour Λ. Next, imagine expanding the contour outward toward infinity. This integral can be replaced by the difference of two integrals, one over a positively oriented circle Λ_∞ of infinite radius, and the other over a positively oriented contour surrounding the branch cut. If one supposes that the appropriate limits exist, the latter integral may be replaced by an integral along the cut $\mathcal{L}$ that involves the discontinuity of the P-potential; viz.,

$$P(\tau) = \frac{1}{2\pi i}\int_{\Lambda_\infty} d\sigma \frac{P(\sigma)}{\sigma - \tau} - \frac{1}{2\pi i}\int_{\mathcal{L}} d\sigma \frac{[P(\sigma)]}{\sigma - \tau}. \quad (36)$$

At $\tau = \infty$, one has

$$P(\infty) = \begin{pmatrix} 0 & \sqrt{2}i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{37}$$

The first integral is also equal to this, and

$$P(\tau) = \begin{pmatrix} 0 & \sqrt{2}i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{2\pi i}\int_{\mathcal{L}} d\sigma \frac{[P(\sigma)]}{\sigma - \tau}. \tag{38}$$

This equation does not tell the whole story, however, for we must make sure that

$$\lim_{t\to 0} F(t) = \Omega, \tag{39}$$

and this requires

$$\sqrt{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\pi}\int_{\mathcal{L}} d\sigma [P_1(\sigma)], \tag{40}$$

where P_1 is the first column of P. The result (38) is valid for τ off the branch cut $\mathcal{L}$. However, we can let τ approach $\mathcal{L}$ from the two sides, and average the results, thereby obtaining, for $\tau \in \mathcal{L}$,

$$\{P(\tau)\} = \begin{pmatrix} 0 & \sqrt{2}i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{2\pi i}\int_{\mathcal{L}} d\sigma \frac{[P(\sigma)]}{\sigma - \tau}. \tag{41}$$

Consider the Hauser-Ernst HHP corresponding to the application of any Kinnersley-Chitre transformation $v(\tau)$ to Minkowski space, i.e.,

$$P(\tau)v(\tau) = Y_+(\tau)P^{MS}(\tau) \tag{42}$$

where $v(\tau)$ and $Y_+(\tau)$ are holomorphic in a neighborhood of $\tau = 0$. We only consider spacetime points such that the branch points $\tau = z \pm i\rho$ of $P^{MS}(\tau)$ and the cut $\mathcal{L}$ from $\tau = z - i\rho$ to $\tau = z + i\rho$ lie within this neighborhood. Thus, as one crosses the branch cut $\mathcal{L}$, $v(\tau)$ and $Y_+(\tau)$ are continuous, but $P(\tau)$ and $P^{MS}(\tau)$ are discontinuous. If $[P(\tau)]$ denotes the discontinuity in $P(\tau)$, and $[P^{MS}(\tau)]$ denotes the discontinuity in $P^{MS}(\tau)$, then, for $\tau \in \mathcal{L}$, we have

$$[P(\tau)]v(\tau) = Y_+(\tau)[P^{MS}(\tau)]. \tag{43}$$

Similarly, if $\{P(\tau)\}$ denotes the average value of $P(\tau)$ across the cut $\mathcal{L}$, and $\{P^{MS}(\tau)\}$ denotes the average value of $P^{MS}(\tau)$, then, for $\tau \in \mathcal{L}$, we have

$$\{P(\tau)\}v(\tau) = Y_+(\tau)\{P^{MS}(\tau)\}. \tag{44}$$

Of course, the values of $[P^{MS}(\tau)]$ and $\{P^{MS}(\tau)\}$ are easily identified using the known form of the P-potential for Minkowski space,

$$P^{MS}(\tau) = \begin{pmatrix} -\frac{[\mu(\tau)+(z-\tau)]}{\sqrt{2}\mu(\tau)} & i\frac{[\mu(\tau)-(z-\tau)]}{\sqrt{2}\mu(\tau)} & 0 \\ -\frac{i}{\sqrt{2}\mu(\tau)} & \frac{1}{\sqrt{2}\mu(\tau)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{45}$$

where

$$\mu(\tau) := \sqrt{(\tau - z)^2 + \rho^2}. \tag{46}$$

Thus, for $\tau \in \mathcal{L}$, we have

$$\begin{aligned} [P^{MS}(\tau)] &= \frac{\sqrt{2}}{\mu(\tau)} \begin{pmatrix} -(z-\tau) & -i(z-\tau) & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, && (47) \\ \{P^{MS}(\tau)\} &= \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, && (48) \end{aligned}$$

and hence

$$\begin{aligned} [P_2^{MS}(\tau)] &= i[P_1^{MS}(\tau)], && (49) \\ \{P_2^{MS}(\tau)\} &= -i\{P_1^{MS}(\tau)\}, && (50) \end{aligned}$$

where the subscripts 1 and 2 refer to the first and second columns, respectively. Whatever may be the form of $Y_+(\tau)$, it is clear, moreover, that

$$\begin{aligned} &[P(\tau)]v_2(\tau) = i[P(\tau)]v_1(\tau), \quad [P(\tau)]v_3(\tau) = 0, && (51) \\ &\{P(\tau)\}v_2(\tau) = -i\{P(\tau)\}v_1(\tau), && (52) \end{aligned}$$

where v_i is the i-th column of v.

If one now specializes to the Sibgatullin transformation (34), one finds

$$\begin{aligned} [P_2(\tau)] &= i\mathcal{E}(\tau, 0)[P_1(\tau)], && (53) \\ [P_3(\tau)] &= \sqrt{2}i\Phi(\tau, 0)[P_1(\tau)], && (54) \end{aligned}$$

while

$$\{P_2(\tau)\} = -i\mathcal{E}(\tau,0)^\dagger\{P_1(\tau)\} - \sqrt{2}\Phi(\tau,0)^\dagger\{P_3(\tau)\}. \tag{55}$$

It is the above equations (53), (54) and (55), which involve directly the axis data, that I find to be particularly appealing about Sibgatullin's formulation.

By considering the separate columns of Eq. (41), we obtain Sibgatullin's integral equation,

$$\sqrt{2}\begin{pmatrix} 1 \\ 0 \\ -i\Phi(\tau,0)^\dagger \end{pmatrix} = \frac{1}{2\pi i}\int_{\mathcal{L}} d\sigma \frac{\mathcal{N}(\sigma,\tau)[P_1(\sigma)]}{\sigma-\tau}, \tag{56}$$

where

$$\mathcal{N}(\sigma,\tau) := \mathcal{E}(\sigma,0) + \mathcal{E}(\tau,0)^\dagger + 2\Phi(\tau,0)^\dagger\Phi(\sigma,0). \tag{57}$$

Sibgatullin selected a branch cut that is a straight line segment. One can then parametrize points on the cut $\mathcal{L}$ so that

$$\tau = z + i\rho x, \quad \sigma = z + i\rho y, \tag{58}$$

where $-1 \leq x \leq +1$ and $-1 \leq y \leq +1$, whereupon one gets

$$\mu(\sigma) = \rho\sqrt{1-y^2}. \tag{59}$$

The discontinuity $[P_1(\sigma)]$ becomes infinite at the branch points, because of a factor $\mu(\sigma)$ in the denominator. For this reason we shall introduce a new column matrix

$$\mathcal{P}(\sigma) := \frac{i\mu(\sigma)}{\sqrt{2}}[P_1(\sigma)], \tag{60}$$

which is continuous and bounded on $\mathcal{L}$. Using Eqs. (58) and (60), we obtain

$$-\rho\begin{pmatrix} 1 \\ 0 \\ -i\Phi(\tau,0)^\dagger \end{pmatrix} = \frac{1}{2\pi}\int_{-1}^{+1} dy \frac{\mathcal{N}(z+i\rho y, z+i\rho x)\mathcal{P}(z+i\rho y)}{\sqrt{1-y^2}(y-x)}. \tag{61}$$

Finally, we have the subsidiary constraint

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\pi}\int_{-1}^{+1} dy \frac{\mathcal{P}(z+i\rho y)}{\sqrt{1-y^2}}, \tag{62}$$

and the complex potentials $\mathcal{E}(z,\rho)$ and $\Phi(z,\rho)$ are given by

$$\begin{aligned}\mathcal{E}(z,\rho) &= \sqrt{2}\lim_{\tau\to\infty}(\tau P_{22}(\tau)) \\ &= \frac{\sqrt{2}}{2\pi i}\int_{\mathcal{L}} d\sigma[P_{22}(\sigma)] \\ &= \frac{\sqrt{2}}{2\pi}\int_{\mathcal{L}} d\sigma\mathcal{E}(\sigma,0)[P_{21}(\sigma)] \\ &= \frac{1}{\pi}\int_{-1}^{+1} dy\frac{\mathcal{E}(z+i\rho y,0)\mathcal{P}_2(z+i\rho y)}{\sqrt{1-y^2}} \qquad (63)\end{aligned}$$

and

$$\Phi(z,\rho) = \frac{1}{\pi}\int_{-1}^{+1} dy\frac{\Phi(z+i\rho y,0)\mathcal{P}_2(z+i\rho y)}{\sqrt{1-y^2}}, \qquad (64)$$

respectively.

5 Concluding remarks

Among the scores of explicit solutions that have been constructed by Vladimir Manko and his colleagues by solving the Sibgatullin integral equation, I don't believe one can find an example that does not start with *rational* axis data. Yet, we know that the Fredholm equation of the second kind, to which the HHP is equivalent, can be solved using traditional methods regardless of whether the axis data are rational or not. Perhaps this apparent limitation of the Sibgatullin approach is an illusion caused by too great a reliance upon inspired guesswork when solving the Sibgatullin integral equation. It would really be nice to see a systematic analysis of the general problem of solving Sibgatullin's integral equation for truly arbitrary axis data, for the Sibgatullin approach to the construction of specific solutions is certainly extremely elegant and efficient.

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